Lecture 17 (5/6)

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We constructed irreducible representations of the Lie algebra $sl_2(\mathbb{C})$:

$$L(n) \quad \begin{tikzcd} \bullet_{v_0} \arrow[r, bend right, "F"'] & \bullet_{v_1} \arrow[l, bend right, "E"'] \arrow[r, bend right, "F"'] & \bullet_{v_2} \arrow[l, bend right, "E"'] & \cdots & \bullet_{v_n} \arrow[l, bend right, "E"'] \arrow[r, "F=0"] & {} \end{tikzcd}$$

and for the Lie group $SL(2,\mathbb{C})$:

$V_n = \{$degree n homogeneous polynomials in $x, y\}$

$= \langle x^n, x^{n-1}y, x^{n-2}y^2, \dots y^n \rangle$

This is the same basis up to scaling, $V_n \simeq L(n)$.

Any complex representation of $sl_2(\mathbb{C}) / SL(2,\mathbb{C})$ is a direct sum of such representations.

Rmk $su_2 \otimes \mathbb{C} = sl_2(\mathbb{C})$, so all irreducible complex representations of su_2 and $SU(2)$ are isomorphic to V_n for some n.

Fact ① Irreducible representations of $so(3)$ are isomorphic to $L(n)$

② A representation $L(n)$ lifts to a representation of Lie group $SO(3)$ if and only if n is even.

Proof ① $so(3) \simeq su(2)$ as Lie algebras, they have same representations.

② $SO(3) \simeq SU(2)/\pm I$, in particular we have a homomorphism $\Phi : SU(2) \longrightarrow SO(3)$

Any representation $\rho : SO(3) \longrightarrow GL(V)$ yields a representation of $SU(2)$: $SU(2) \xrightarrow{\Phi} SO(3) \xrightarrow{f} GL(V)$

Conversely, given a representation of $SO(2)$, we get a representation of $SO(3)$ iff $\pm I$ acts <u>trivially</u>.

$$V_n = \langle x^n, x^{n-1}y, \dots, y^n \rangle$$

$-I \in SU(2)$ acts on V_n diagonally: $x \to -x$, $y \to -y$

$$x^n \longrightarrow (-1)^n x^n$$
$$x^{n-1}y \longrightarrow (-1)^n x^{n-1} y$$
$$\dots$$

$-I$ acts by $(-1)^n I$.

$\Rightarrow$ $-I$ acts trivially iff <u>n even</u>.

<u>Cor</u> Irred. representations of $SO(3;\mathbb{R})$ are

$V_0 = \langle 1 \rangle$ trivial rep. $\quad V_2 = \langle x^2, xy, y^2 \rangle$ = defining representation of $SO(3)$.

$\dim V_4 = 5$, $\dim V_6 = 7, \dots$

<u>Rmk</u> Related to spectrum of hydrogen/quantum mechanics.

Operations on representations

G = Lie group $\quad \mathfrak{g} = \mathrm{Lie}(G)$ Lie algebra

U, V = representations

- Direct sum: $U \oplus V$, $\dim U \oplus V = \dim U + \dim V$
- Tensor product: $U \otimes V$ tensor product of underlying vector spaces

$$\dim U \otimes V = \dim U \cdot \dim V$$

$A \in G$ acts on $u \otimes v \in U \otimes V$

by $A(u \otimes v) = (Au) \otimes (Av)$.

— product

by $A(u\otimes v) = (Au)\otimes(Av)$.

$X \in \mathfrak{g}$ acts on $u \otimes v$:

$$X(u\otimes v) = (Xu)\otimes v + u\otimes(Xv)$$ product rule for derivatives.

In particular, if $Xu = \lambda u$, $Xv = \mu v$ then

$$X(u\otimes v) = (\lambda+\mu)\, u\otimes v.$$

Characters V = complex representation of $\mathfrak{sl}_2(\mathbb{C})$

Recall: H diagonalizes, eigenvalues of H are integers.

a_i = dimension of eigenspace for H with eigenvalue i

$$\operatorname{ch} V = \sum_i a_i q^i \quad \Leftarrow \text{Laurent polynomial in } q$$

$$\operatorname{ch} L(n) = q^n + q^{n-2} + \dots + q^{-n}$$

Fact $\operatorname{ch}(U\oplus V) = \operatorname{ch}(U) + \operatorname{ch}(V)$

$\operatorname{ch}(U\otimes V) = \operatorname{ch}(U)\cdot\operatorname{ch}(V)$

From Lie group perspective, we can consider

$$e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$ On an H-eigenspace with eigenvalue k, e^{tH} acts by e^{tk}

Define $q = e^t$

$$e^{tH} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$ acts on H-eigenspace with eigenvalue k by q^k

Conclusion $\boxed{\operatorname{ch}(V) = \operatorname{Tr}_V \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.}$

element of $SL(2;\mathbb{C})$

$g \longrightarrow \operatorname{Tr}_V(g)$ is a function on G

We just restrict this function to {diagonal, ...} $\subset G$.

matrices)

$V_2 = L(2) = \langle x^2, xy, y^2 \rangle \qquad \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (qx, q^{-1}y)$

$\operatorname{ch} V_2 = q^2 + 1 + q^{-2}$

$\operatorname{ch} V_2 \otimes V_2 = (q^2 + 1 + q^{-2})(q^2 + 1 + q^{-2}) =$

$= q^4 + 2q^2 + 3 + 2q^{-2} + q^{-4} \quad (*)$

How to decompose $V_2 \otimes V_2$ into irreducibles?

$$(*) = \underbrace{(q^4 + q^2 + 1 + q^{-2} + q^{-4})}_{= \operatorname{ch} V_4} + \underbrace{(q^2 + 1 + q^{-2})}_{= \operatorname{ch} V_2} + \underbrace{1}_{= \operatorname{ch} V_0}$$

$$\boxed{V_2 \otimes V_2 \simeq V_4 \oplus V_2 \oplus V_0}$$

$\boxed{\begin{array}{l} \text{HW 6: } V_a \otimes V_b = ? \\ V_1 \otimes V_1 \otimes V_1 = ? \end{array}}$

Rmk $V_2 \otimes V_2 = \operatorname{Sym}^2(V_2) \oplus \Lambda^2(V_2)$

- $V_2 \otimes V_2 \simeq$ 3×3 matrices $\circlearrowleft SO(3)$
- $\operatorname{Sym}^2(V_2) \simeq$ symmetric 3×3 matrices, $\dim = 6$
- $\Lambda^2(V_2) \simeq$ skew-symmetric 3×3 matrices, $\dim = 3$

By decomposition above, we see that $\Lambda^2 V_2 \cong V_2$

$\operatorname{Sym}^2(V_2) \simeq V_4 \oplus V_0$

$SO(3)$ acts on the space of symmetric 3×3 matrices preserving the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which spans a trivial representation in $\operatorname{Sym}^2(V_2)$.