Root systems

$\mathfrak{g}$ = Lie algebra

Recall: adjoint representation
$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$
$\mathrm{ad}_X(Z) = [X, Z]$

① Killing form on $\mathfrak{g}$

$$(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$$

Fact: For $\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{sp}_n$ etc

$$(X,Y) = \underbrace{\text{factor}}\,\mathrm{Tr}(XY)$$

Key definition $\mathfrak{g}$ is <u>semisimple</u> if $(\,,\,)$ is nondegenerate

Fact If $\mathfrak{g}$ is semisimple then the center of $\mathfrak{g}$ is trivial (center = $\{X$ such that $[X,Z]=0$ for all $Z\}$)

Proof If X is in the center of $\mathfrak{g}$ then $\mathrm{ad}_X = 0$
$\Rightarrow (X,Y) = 0$ for all Y, contradiction

We will identify $\mathfrak{g} \simeq \mathfrak{g}^*$ via the Killing form.

Ex $\mathfrak{sl}_2$ $\quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\underset{\mathrm{Tr}(EF)}{\underset{\shortparallel}{(E,F)}} = (F,E) = 1 \qquad \underset{\mathrm{Tr}(H^2)}{\underset{\shortparallel}{(H,H)}} = 2 \qquad \text{and all other } (\,,\,) = 0$$

Goal: understand the structure of semisimple Lie algebras.

② **Key example**: $\mathfrak{sl}_3$

Basis: $H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $\quad H_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ ← diagonal

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$E_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ $E_{12} = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} = [E_1, E_2]$ ← upper triangular

$F_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$ $F_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}$ $F_{12} = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix} = -[F_1, F_2]$ ← lower triangular

Some relations

- $[H_1, H_2] = 0$, so H_1, H_2 span a commutative 2d Lie subalgebra (Cartan subalgebra)
- $[H_1, E_1] = 2E_1$ $[H_1, E_2] = -E_2$ (check it!)
  $[H_2, E_1] = -E_1$. $[H_2, E_2] = 2E_2$
- H_1, E_1, F_1 span a copy of sl_2 inside sl_3!
  H_2, E_2, F_2 span a copy of sl_2
  $H_1 + H_2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$, E_{12}, F_{12} span a copy of sl_2.

③ Def $\mathfrak{g}$ = semisimple Lie algebra.

A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a maximal subspace such that:

- $[X_1, X_2] = 0$ for $X_1, X_2 \in \mathfrak{h}$ (commutative)
- ad_X is diagonalizable on $\mathfrak{g}$ for

all $x \in \mathfrak{h}$.

__Fact__ Cartan subalgebras exist.

__Ex__ $\mathfrak{g} = \mathfrak{sl}_2 \quad \mathfrak{h} = \langle H \rangle \qquad ad_H(E) = [H, E] = 2E$

$ad_H(H) = 0 \qquad ad_H(F) = [H, F] = -2F$

So E, F, H are eigenvectors for ad_H.
and ad_H is diagonalizable.

__Ex__ $\mathfrak{g} = \mathfrak{sl}_3 \quad \mathfrak{h} = \langle H_1, H_2 \rangle$

$ad_{H_1}(H_1) = ad_{H_1}(H_2) = 0$

$ad_{H_1}(E_1) = 2E_1 \qquad ad_{H_1}(E_2) = [H_1, E_2] = -E_2$

$ad_{H_1}(E_{12}) = E_{12}$

$ad_{H_1}(F_1) = -2F_1 \qquad ad_{H_1}(F_2) = F_2$

$ad_{H_1}(F_{12}) = -F_{12}$.

So: ad_{H_1} is diagonalizable in our basis!
Similarly ad_{H_2} is diagonalizable in our basis.

(4) Very important Lemma.

Suppose $A_1 \ldots A_k$ are diagonalizable operators on a f.dim vector space V.
If $[A_i, A_j] = 0$ for all i, j then there is a common eigenbasis in V for all A_i.

Basis $v_1 \ldots v_n$, $A_j v_m = \lambda_{jm} v_m$ for all j, m

<u>Proof</u>: First, diagonalize A_1, let λ be some eigenvalue for A_1. Consider the eigenspace $V_\lambda = \{v : A_1 v = \lambda v\}$

We claim that $A_2 \ldots A_k$ preserve V_λ. Indeed, if $v \in V_\lambda$, $A_1(A_j v) = A_j(A_1 v) = A_j(\lambda v) = \lambda \cdot A_j v$

so $A_j v$ is also in V_λ.

Then we can diagonalize A_2 on V_λ, and continue by induction. ∎

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⑤ As a consequence of Very Important Lemma, we can <u>diagonalize</u> ad_X for all $X \in \mathfrak{h}$ <u>simultaneously</u>.

<u>Ex</u> sl_3 $\quad [H_1, E_1] = 2E_1 \qquad [H_2, E_1] = -E_1$

$X \in \mathfrak{h} \quad X = aH_1 + bH_2$ for some a, b

$$ad_X(E_1) = [X, E_1] = [aH_1 + bH_2, E_1] = a \cdot 2E_1 + b(-E_1) = (2a - b)E_1 .$$

So E_1 is an eigenvector for any element X in the Cartan subalgebra, but the eigenvalue depends on X.

In fact, the <u>eigenvalue</u> is a linear functional on $\mathfrak{h}$! (for fixed eigenvector)

<u>Observe</u> $\boxed{ad_X(E_1) = (X, H_1)E_1}$ so in this case the eigenvalue of X is given by (X, H_1).

<u>Proof</u> Both sides are linear in X, so need to check for $X = H_1$ or $X = H_2$.

$$ad_{H_1}(E_1) = 2E_1 \qquad ad_{H_2}(E_1) = -E_1$$

$$(H_1, H_1) = 2 \qquad (H_1, H_2) = -1. \qquad \blacksquare$$

Similarly, $ad_X(E_2) = (X, H_2)E_2$

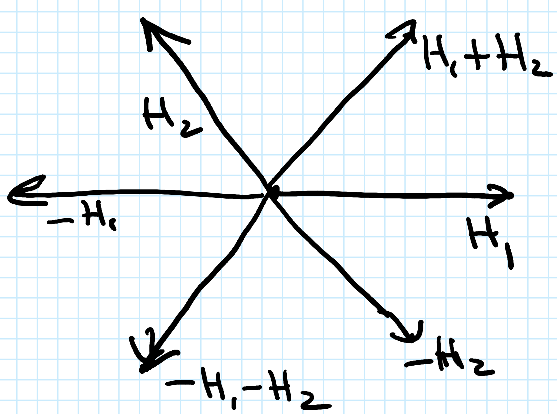
$ad_X(E_{12}) = (X, H_1 + H_2)E_{12}$

<u>Picture:</u> draw $\mathfrak{h} = \mathrm{Span}(H_1, H_2)$ as 2d vector space w. Killing form

$\mathrm{length}(H_1) = \sqrt{2}$ since $(H_1, H_1) = 2$
$\parallel$
$\mathrm{length}(H_2)$

$$(H_1, H_2) = \sqrt{2} \cdot \sqrt{2} \cdot \cos(\varphi) = -1$$

$$\cos(\varphi) = -\frac{1}{2} \qquad \varphi = \frac{2\pi}{3}$$



<u>Claim</u> By the above, this picture contains all the information about ad_X for X in Cartan subalgebra.