Lecture 19 (5/11)

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Recap $\mathfrak{g}$ = Lie algebra

$(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) \sim \mathrm{Tr}(XY)$ in nice examples Killing form

$\mathfrak{g}$ semisimple if $(\,,\,)$ nondegenerate

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra maximal subalgebra such that

1) $[X_1, X_2] = 0$ for $X_1, X_2 \in \mathfrak{h}$ commutative

2) ad_X diagonalizable for $X \in \mathfrak{h}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \qquad \alpha = \text{roots} \in \mathfrak{h}$$

$\mathfrak{g}_\alpha = \{ Y : \mathrm{ad}_X(Y) = [X, Y] = (\alpha, X) Y$ for all $X \in \mathfrak{h} \}$

root subspace = eigenspace for ad_X

common eigenspace for all $X \in \mathfrak{h}$

Ex $\mathfrak{sl}_n$

$$Y = \begin{array}{c} \begin{pmatrix} 0 & & & & \\ & 0 & \cdots & 1 & \\ & & \ddots & \vdots & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \leftarrow i \\ \uparrow j \end{array} \qquad X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$$

X = arbitrary diagonal matrix

$[X, Y] = (x_i - x_j) Y$

Note $x_i - x_j = (X, \alpha_{ij})$ where

$$\alpha_{ij} = \begin{array}{c} \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Y spans a root subspace for the root α_{ij}

$$\boxed{\mathfrak{g}_{\alpha_{ij}} = \mathrm{Span}(Y)}$$

In fact, α_{ij} for all $i \neq j$ are roots of $\mathfrak{sl}_n$.

Summary We encode the info about $\mathfrak{g}$ using a finite collection of roots $\alpha \in \mathfrak{h}$.

Lemma 1 $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

Pf $X \in \mathfrak{h}$, $Y \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_\beta$

$$[X,[Y,Z]] \underset{\text{Jacobi identity}}{=} [[X,Y],Z] + [Y,[X,Z]] =$$

$$= (\alpha, X)[Y,Z] + (\beta, X)[Y,Z]$$
$$= (\alpha+\beta, X)[Y,Z]$$

So $[Y,Z] \in \mathfrak{g}_{\alpha+\beta}$. ∎

Lemma 2 If $\alpha+\beta \neq 0$ then $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ wrt Killing form.

Proof $Y \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_\beta$ $\quad (Y,Z) = Tr(ad_Y\, ad_Z) \Leftarrow$ want to show $=0$

$$ad_Y\, ad_Z(\mathfrak{g}_\gamma) \underset{\text{Lemma 1}}{\subset} ad_Y(\mathfrak{g}_{\beta+\gamma}) \underset{\text{Lemma 1}}{\subset} \mathfrak{g}_{\alpha+\beta+\gamma}$$

$ad_Y\, ad_Z : \mathfrak{g}_\gamma \longrightarrow \mathfrak{g}_{\alpha+\beta+\gamma}$ $\quad$ if $\alpha+\beta\neq 0$ then $\gamma \neq \alpha+\beta+\gamma$

Therefore $Tr(ad_Y\, ad_Z) = 0$. ∎

$\boxed{\text{Corollary}}$ $\boxed{\mathfrak{h} \perp \mathfrak{g}_\alpha \text{ for all } \alpha}$

Lemma 3 If α is a root then $-\alpha$ is a root.

Proof Assume $-\alpha$ is not a root, then $\alpha+\beta\neq 0$ for all roots β

By Lemma 2, this means $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all β

$\Rightarrow$ the form is degenerate. Contradiction ∎

Lemma 4 The roots span $\mathfrak{h}$.

Proof Assume not, pick $X \in \mathfrak{h}$ such that $(X,\alpha) = 0$ for all α.

For $Y \in \mathfrak{g}_\alpha$ we get $[X,Y] = (X,\alpha)Y = 0$

For $X' \in \mathfrak{h}$ we have $[X, X'] = 0$

so X commutes with any element of $\mathfrak{g}$ $\Rightarrow \mathrm{ad}_X = 0$

$\Rightarrow (,)$ degenerate. Contradiction.

<u>Thm</u> For all α there exist an <u>sl_2-triple</u>

$(E_\alpha, F_\alpha, H_\alpha)$ where $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{h}$

In fact, starting from any $E \in \mathfrak{g}_\alpha$ we can complete it to an sl_2-triple with $E = E_\alpha$.

And $(E_\alpha, F_\alpha, H_\alpha) \simeq sl_2$ as Lie algebra.

<u>Proof</u> Pick $E \in \mathfrak{g}_\alpha$. Since $(,)$ is nondegenerate and $E \perp \mathfrak{g}_\beta$ for $\beta \neq -\alpha$, we can choose a vector $F \in \mathfrak{g}_{-\alpha}$ such that $(E, F) = 1$.

Observe that $[E, F] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{h}$ since $\alpha + (-\alpha) = 0$

We want to identify $[E, F]$ in the Cartan. Pick $X \in \mathfrak{h}$

$$([E,F], X) = \mathrm{Tr}(\mathrm{ad}_{[E,F]} \mathrm{ad}_X) = \mathrm{Tr}([\mathrm{ad}_E, \mathrm{ad}_F]\mathrm{ad}_X)$$

$$= \mathrm{Tr}(\mathrm{ad}_E \mathrm{ad}_F \mathrm{ad}_X - \mathrm{ad}_F \mathrm{ad}_E \mathrm{ad}_X) =$$

$$= \mathrm{Tr}(\mathrm{ad}_E \mathrm{ad}_F \mathrm{ad}_X - \mathrm{ad}_E \mathrm{ad}_X \mathrm{ad}_F) = \mathrm{Tr}(\mathrm{ad}_E [\mathrm{ad}_F, \mathrm{ad}_X])$$

$$= (E, [F, X]) \qquad \boxed{([E,F], X) = (E, [F, X])}$$

Now $(E, [F, X]) = -(E, [X, F]) \underset{\substack{\text{since} \\ F \in \mathfrak{g}_{-\alpha}}}{=} -(E, (-\alpha, X) F) =$

$$= (\alpha, X) \cdot (E, F) = (\alpha, X) \quad \text{since } (E, F) = 1.$$

Conclusion: $([E, F], X) = (\alpha, X)$ for all $X \in \mathfrak{h}$

Since $(,)$ is nondegenerate, we get $[E, F] = \alpha$ on Cartan

<u>Define</u> $E_\alpha = E$, $F_\alpha = \frac{2}{(\alpha,\alpha)} F$, $H_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$

Let us prove that this is an sl_2-triple:

$$[\underbrace{H_\alpha}_{\text{in } \mathfrak{h}}, \underbrace{E_\alpha}_{\mathfrak{g}_\alpha}] = (H_\alpha, \alpha) E_\alpha = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)} E_\alpha = 2E_\alpha$$

$$[H_\alpha, F_\alpha] = (H_\alpha, -\alpha) F_\alpha = -\frac{2(\alpha,\alpha)}{(\alpha,\alpha)} F_\alpha = -2F_\alpha$$

$$[E_\alpha, F_\alpha] = [E, \frac{2}{(\alpha,\alpha)} F] = \frac{2}{(\alpha,\alpha)} [E, F] = \frac{2\alpha}{(\alpha,\alpha)} = H_\alpha .$$

∎

$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_\alpha$ for $\alpha = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad (E, F) = Tr(EF) = 1.$

$[E, F] = \alpha$ $\quad$ In this case $(\alpha,\alpha) = 2$

$\frac{2}{(\alpha,\alpha)} = 1 \Rightarrow E, F, \alpha$ form an sl_2-triple.