A matrix Lie group = closed subgroup $G \subset GL(n, \mathbb{C})$ of $GL(n, \mathbb{C})$

- subgroup
- If $A_n \in G$, $\lim_{n\to\infty} A_n = A$ then either $A \in G$ or A is not invertible

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Ex: $GL(n, \mathbb{R})$, $GL^+(n, \mathbb{R}) = \{A : \det A > 0\}$

A_n real $\Rightarrow$ A real $\quad$ $\lim_{n\to\infty} A_n = A$ $\quad$ $\det A_n > 0 \Rightarrow \det A \geq 0$

$SL(n, \mathbb{R}) = \{A : \det A = 1\}$, $SL(n, \mathbb{C})$.

Ex Any finite subgroup of $GL(n, \mathbb{C})$ is a matrix Lie group

Ex $GL(n, \mathbb{Q})$ is NOT a matrix Lie group (not closed)

Ex $\{2^k, k \in \mathbb{Z}\}$ is a matrix Lie group in $GL(1)$

$\lim_{k\to -\infty} 2^k = 0$ but this is not invertible.

The orthogonal group $O(n)$

Def A is orthogonal if $A^T A = I$

Fact This is equivalent to:

① Columns of A form an orthonormal basis

② $\forall x, y \in \mathbb{R}^n$ $(Ax, Ay) = (x, y)$ where $\sum x_i y_i = (x, y)$ is the dot product

Note: $(x,y) = x^T y$, $(Ax, Ay) = (Ax)^T (Ay) = x^T A^T A y$

③ A sends any orthonormal basis to an orthonormal basis,

Lemma: a) A orthogonal $\Rightarrow (\det A)^2 = \det A^T \cdot \det A = 1$
$\Rightarrow \det A = \pm 1$

b) A orthogonal $\Rightarrow A$ invertible and $A^{-1} = A^T$ is orthogonal

c) A, B orthogonal $\Rightarrow AB$ is orthogonal.

Cor $O(n)$ is a matrix Lie group.

Thm $O(n)$ is compact

Pf: The equation A^TA defines a closed subset of $\mathrm{Mat}(n\times n) = \mathbb{R}^{n^2}$

- It is bounded: since all columns have length 1 $\Rightarrow |a_{ij}| \leq 1$

So $O(n)$ is closed and bounded in $\mathbb{R}^{n^2} \Rightarrow$ compact. ∎

Ex $O(2)$: $\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ $|v_1| = 1 \Rightarrow v_1 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$v_2 \perp v_1 \Rightarrow v_2 = k \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$

$|v_2| = 1 \Rightarrow k = \pm 1$ $\qquad 0 \leq \varphi \leq 2\pi$

So: $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ or $\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$

det = 1, rotation $\qquad$ det = −1, reflection

$\boxed{O(2) = S^1 \sqcup S^1}$

Thm $O(n)$ has 2 connected components for all n.

Pf: $\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} = A$ $\quad a_i$ = orthonormal basis

Choose a 2-plane P containing a_1 and $e_1 = (1, 0, \ldots, 0)$

$\exists$ orthogonal transform: • rotation in P

(G can rotate continuously) • Id on $P^\perp$ which sends a_1 to e_1

Now we changed our orthonormal basis to

$$\begin{pmatrix} 1 & 0 \; \cdots \; 0 \\ 0 & \\ \vdots & \boxed{} \\ 0 & \end{pmatrix}$$ and proceed by induction

until $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & & & ? \end{pmatrix}$

Since we still have an orthonormal basis, $? = \pm 1$.

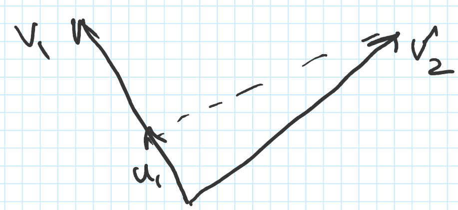
To sum up, any orthogonal matrix can be connected to $\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$ by a continuous path. $\blacksquare$

Thm $GL(n,\mathbb{R})$ retracts onto $O(n)$

Proof Gram–Schmidt process.

A matrix in $GL(n) \longleftrightarrow$ basis $v_1 \ldots v_n$ of columns

- $u_1 = \dfrac{v_1}{|v_1|}$



- $v_2' = v_2 \underbrace{- (v_2, u_1) u_1}_{\text{projection of } v_2 \text{ onto } u_1}$

$(v_2', u_1) = (v_2, u_1) - (v_2, u_1) \underbrace{(u_1, u_1)}_{1} = 0$

$u_2 = \dfrac{v_2'}{|v_2'|}$

- $v_3' = v_3 - (v_3, u_1) u_1 - (v_3, u_2) u_2$

$u_3 = \dfrac{v_3'}{|v_3'|}$ and so on.

Thus, starting from any basis $v_1 \ldots v_n$ we get an orthonormal basis $u_1 - - u_n$.

Note: we can do all steps continuously:

— $v_i' \longrightarrow t v_i' + (1-t)\frac{v_i'}{|v_i|} = v_i'\underbrace{\left(t + (1-t)\frac{1}{|v_i|}\right)}_{>0}$

at $t=1$ we get v_i'

at $t=0$ we get $\frac{v_i'}{|v_i|} = u_i$

— $v_i \longrightarrow v_i - t\sum_{j<i}(v_i, u_j)\, u_j$

at $t=0$ we get v_i

at $t=1$ we get v_i'

As a result, we get a sequence of retractions which compose to a retraction from $GL(n,\mathbb{R})$ to $O(n)$. ∎

Cor. • $GL(n,\mathbb{R})$ is homotopy equivalent to $O(n)$

• $GL(n,\mathbb{R})$ has exactly 2 connected components.