Lecture 20 (5/13)

Friday, May 13, 2022 12:41 PM

Recap $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$ α = roots

$\mathfrak{g}_\alpha$ — common eigenspaces for Cartan subalgebra

Thm (last lecture) For any $E \in \mathfrak{g}_\alpha$ there is an sl_2-triple $(E = E_\alpha, H_\alpha, F_\alpha)$ such that $H_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$, $F_\alpha \in \mathfrak{g}_{-\alpha}$.

Key idea today: This sl_2 acts on $\mathfrak{g}$ by $ad_{E_\alpha}, ad_{H_\alpha}, ad_{F_\alpha}$

So $\mathfrak{g}$ = representation of sl_2, and we can use sl_2 rep. theory to understand the structure of $\mathfrak{g}$.

Thm 1 For any roots α, β we have $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$!

Proof Pick $Y \in \mathfrak{g}_\beta$, it is an eigenvector for H_α:

$$[H_\alpha, Y] = (\beta, H_\alpha) Y = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} Y$$

since $H_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$.

Y is an eigenvector for $H_\alpha \in sl_2$ with eigenvalue $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, but for a finite dim. representation of sl_2, all eigenvalues are integers ∎

$$\{\mathbb{Q}\text{-span of roots}\} \subset \{\mathbb{R}\text{-span of roots}\} \subset \{\mathbb{C}\text{-span of roots}\}$$

can draw ($\mathbb{R}$-span of roots)

$\mathbb{C}$-span of roots $= \mathfrak{h}$

Thm (a) The Killing form on $\mathbb{Q}$-span of roots is $\mathbb{Q}$-valued and positive definite

(b) The Killing form on $\mathbb{R}$-span of roots is positive definite.

Rmk The Killing form on $\mathfrak{g}$ does not need to be positive def.
(it is not for $\mathfrak{sl}_2(\mathbb{R})$).

Proof: Pick X, X' from the Cartan subalgebra

$(X,X') = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_{X'})$, ad_X and $\mathrm{ad}_{X'}$ are diagonalizable with common eigenspaces $\mathfrak{g}_\alpha$

$$= \sum_\alpha (X,\alpha)(X',\alpha) \dim \mathfrak{g}_\alpha$$

$\mathrm{ad}_X \mathrm{ad}_{X'}$ = product of two diagonal matrices.

Pick a root β, then plug in $X = X' = \beta$

$$(\beta,\beta) = \sum_\alpha (\beta,\alpha)(\beta,\alpha) \dim \mathfrak{g}_\alpha = \sum_\alpha (\beta,\alpha)^2 \dim \mathfrak{g}_\alpha$$

divide by $(\beta,\beta)^2$:

$$\frac{1}{(\beta,\beta)} = \sum_\alpha \frac{(\beta,\alpha)^2}{(\beta,\beta)^2} \dim \mathfrak{g}_\alpha.$$

By Thm 1, $\frac{2(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$, $\frac{(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Q}$ $\Rightarrow$ $(\beta,\beta) > 0$ and $(\beta,\beta) \in \mathbb{Q}$.

Therefore $(\beta,\alpha) = \frac{(\beta,\alpha)}{(\beta,\beta)} \cdot (\beta,\beta) \in \mathbb{Q}$.

For any $X \in \mathbb{Q}$-span (roots)

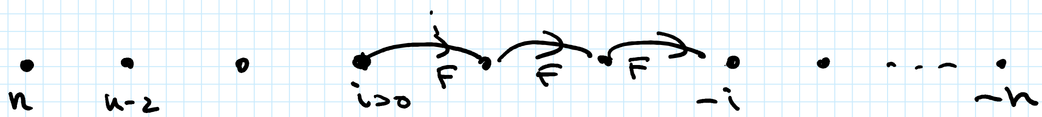
$$(X,X) = \sum_\alpha (X,\alpha)^2 \dim \mathfrak{g}_\alpha > 0 \quad \text{since } (X,\alpha) \in \mathbb{Q}.$$

Lemma Suppose V is finite dim representation of $\mathfrak{sl}_2$
V_i = eigenspace for H with eigenvalue i

(a) If $i \geq 0$ then $F^i : V_i \longrightarrow V_{-i}$ is an isomorphism

(b) If $i \leq 0$ then $E^{|i|} : V_i \longrightarrow V_{-i}$ is an isomorphism.

Proof $V = \bigoplus L(u_i)$, sufficient to check for $L(u)$



$\blacksquare$

Thm 3 Suppose that α, β are roots. Then

$$s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \text{ is also a root.}$$

Proof $(E_\alpha, H_\alpha, F_\alpha) = sl_2$ triple, choose $Y \in \mathfrak{g}_\beta$

Y is an eigenvector for H_α w. eigenvalue $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = i$.

Assume $i > 0$, then by Lemma $F_\alpha^i Y \neq 0$

On the other hand, $F_\alpha^i Y \in \mathfrak{g}_{\beta - i\alpha} \Rightarrow$ $\beta - i\alpha$ is a root.

$F_\alpha \in \mathfrak{g}_{-\alpha}$ $\qquad F_\alpha^i Y = (\mathrm{ad}_{F_\alpha})^i Y$

The proof for $i < 0$ similar. $\blacksquare$

Thm 4 1) If β is a root collinear to α ($\beta = k\alpha$) then $\beta = \pm\alpha$

2) $\dim \mathfrak{g}_\alpha = 1$ for any root α.

Proof: Thm 7.23 in the book.

Def An (abstract) root system is a finite collection of vectors in $\mathbb{R}^n$ w. positive definite form ($=$ dot product)

Satisfying the following conditions:

1. The roots span $\mathbb{R}^n$
2. If α is a root then $-\alpha$ is a root, and the only multiples of α are $\pm\alpha$
3. $\frac{2(\alpha,\beta)}{} \in \mathbb{Z}$ for all roots α, β

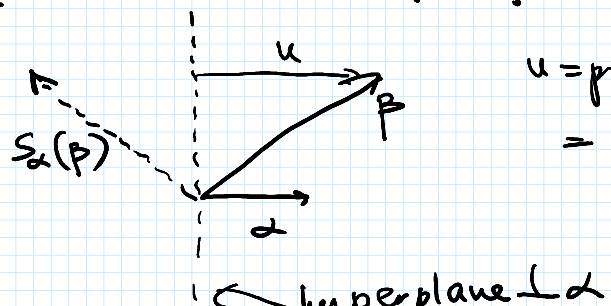
(α,α)

④ If α, β are roots then

$$s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \quad \text{is also a root.}$$

Summary of this week: Starting from a complex semisimple Lie algebra $\mathfrak{g}$, we can construct a root system satisfying ①–④.

Q: What does $s_\alpha(\beta)$ from ④ mean?

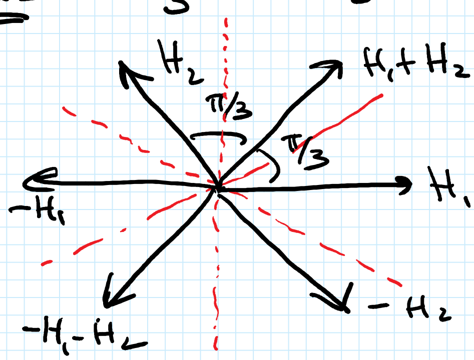


u = projection of β on α

$$= \frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha$$

$$\boxed{s_\alpha(\beta) = \beta - 2u = \text{reflection of } \beta \text{ in the hyperplane perpendicular to } \alpha}$$

Ex 1 $\mathfrak{sl}_3 \leftrightarrow A_2$ root system



all vectors have same length ℓ

$$\frac{2(H_1,H_2)}{(H_1,H_1)} = \frac{2\cdot\ell\cdot\ell\cdot\cos\left(\frac{2\pi}{3}\right)}{\ell\cdot\ell} =$$

$$= 2\cos\left(\frac{2\pi}{3}\right) = 2\cdot\left(-\frac{1}{2}\right) = -1.$$

integer.

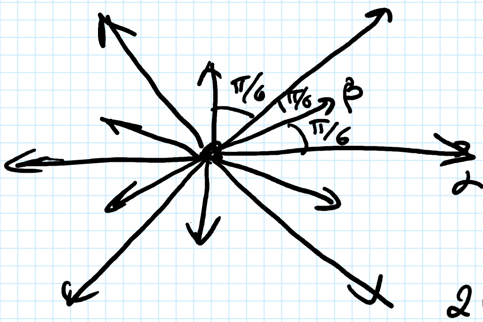
If angle is $\frac{\pi}{3}$, get $+1$.

Red lines perpendicular to roots, reflections in them preserve the root system.

Ex 2 G_2 root system

long roots have length ℓ

short roots have length $\frac{\ell}{\sqrt{3}}$



short roots have length $\frac{l}{\sqrt{3}}$

angles = multiples of $\frac{\pi}{6}$

α long, β short, angle $= \frac{\pi}{6}$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2\cdot l\cdot \frac{l}{\sqrt{3}}\cdot\cos\left(\frac{\pi}{6}\right)}{l\cdot l} =$$

$$= \frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = 1 \quad (!)$$

integers

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{2\cdot l\cdot\frac{l}{\sqrt{3}}\cdot\cos\left(\frac{\pi}{6}\right)}{\frac{l}{\sqrt{3}}\cdot\frac{l}{\sqrt{3}}} = \frac{2}{\sqrt{3}}\cdot 3\cdot\cos\left(\frac{\pi}{6}\right) = 3$$