Root systems

Def A root system of rank n is a collection of vectors in $\mathbb{R}^n$ such that:

1. Root span $\mathbb{R}^n$
2. If α is a root then $(-\alpha)$ is a root and these are the only multiples of α in this collection
3. If α, β are roots then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
4. $S_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is also a root if α, β are roots.

Rmk $S_\alpha(\beta)$ = reflection of β in the hyperplane perpendicular to α

Main thm from last week: $\mathfrak{g}$ = semisimple Lie algebra (complex) $\Longrightarrow$ root system.

Def Weyl group W = group generated by reflections S_α as above.

(4) $\Rightarrow$ W permutes the roots $\Longrightarrow$ W is a finite group (subset of all permutations of roots).

Key example $\mathfrak{sl}_n \Longrightarrow$ root system of type A_{n-1}

$\mathbb{R}^{n-1}$ = all vectors $(x_1 \ldots x_n)$: $\sum x_i = 0$

Roots $= (0 \ldots 0 \overset{i}{1} 0 \ldots 0 \overset{j}{-1} 0 \ldots 0)$ (= digonal matrices in $\mathfrak{sl}_n$)

$(0 \ldots 0 \, {-1} \, 0 \ldots 0 \, 1 \, 0 \ldots 0)$

$2\binom{n}{2}$ roots.

$(\alpha, \alpha) = 2$ for all roots here (all have length $\sqrt{2}$)

$(\alpha, \beta) \in \{\pm 2, 1, -1, 0\} \implies \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = (\alpha, \beta) \in \mathbb{Z}$.

Hyperplane $\perp \underbrace{(0 \ldots 0 \overset{i}{1} 0 \ldots 0 \overset{j}{-1} 0 \ldots 0)}_{\alpha} = \{x_i - x_j = 0\}$

Reflection in such hyperplane $= S_\alpha(x_1 \ldots x_n) = (x_1 \ldots x_j \ldots x_i \ldots x_n)$ swap i and j

S_α = transposition $(i\,j)$

W = group generated by $S_\alpha = S_n$.

Lemma Suppose α, β roots, $\alpha \neq \pm\beta$, $(\alpha,\alpha) \geq (\beta, \beta)$

Then we have the following cases: φ = angle between α and β

(a) $(\alpha, \beta) = 0$ $\quad \alpha \perp \beta$

(b) $(\alpha,\alpha) = (\beta,\beta)$ and $\varphi = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ $\quad \Big| \; \ell(\alpha) = \ell(\beta)$

(c) $(\alpha,\alpha) = 2(\beta,\beta)$ and $\varphi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ $\quad \Big| \; \ell(\alpha) = \sqrt{2}\,\ell(\beta)$

(d) $(\alpha,\alpha) = 3(\beta,\beta)$ and $\varphi = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. $\quad \Big| \; \ell(\alpha) = \sqrt{3}\,\ell(\beta)$.

Proof $m_1 = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \qquad m_2 = \frac{2(\alpha,\beta)}{(\beta,\beta)} \qquad m_1, m_2 \in \mathbb{Z}$

$$0 \leq m_1 m_2 = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\varphi \leq 4$$

integer!

- $m_1 m_2 = 0 \implies (\alpha, \beta) = 0$ ✓
- $m_1 m_2 = 4 \implies \cos^2\varphi = 1 \Rightarrow \cos\varphi = \pm 1 \Rightarrow \alpha = \pm\beta$ not allowed
- $m_1 M_2 = 1 \implies 4\cos^2\varphi = 1 \Rightarrow \cos^2\varphi = \frac{1}{4} \Rightarrow \cos\varphi = \pm\frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$m_1 \quad (\ldots)$

$$\frac{m_2}{m_1} = \frac{(\alpha,\alpha)}{(\beta,\beta)} \geq 1 \Rightarrow (\alpha,\alpha) = (\beta,\beta) \text{ in this case.}$$

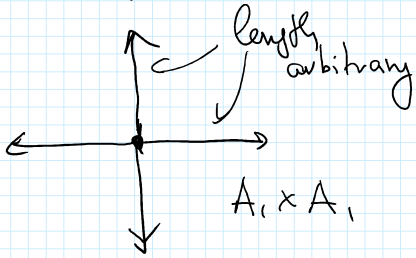
since we assume $(\alpha,\alpha) \geq (\beta,\beta)$

- $m_1 m_2 = 2 \Rightarrow 4\cos^2\varphi = 2 \Rightarrow \cos^2\varphi = \frac{1}{2} \Rightarrow \cos\varphi = \pm\frac{1}{\sqrt{2}} \Rightarrow \varphi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

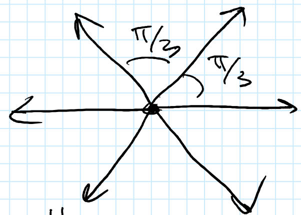
  $m_1 = \pm 1 \quad m_2 = \pm 2 \qquad \frac{m_2}{m_1} = \frac{(\alpha,\alpha)}{(\beta,\beta)} = 2$

- $m_1 m_2 = 3 \Rightarrow 4\cos^2\varphi = 3 \Rightarrow \cos^2\varphi = \frac{3}{4} \Rightarrow \cos\varphi = \pm\frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

All possible root systems in rank 2:

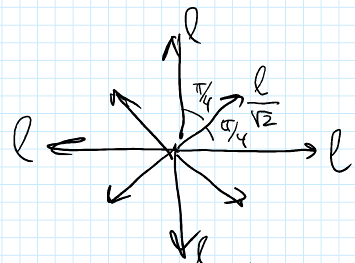


$A_1 \times A_1$



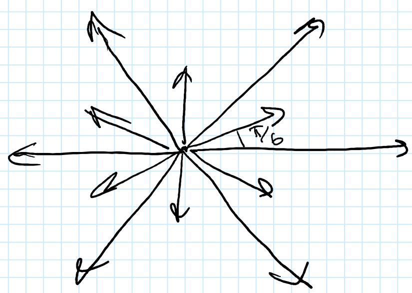
A_2

all have same length



B_2

all short $= l/\sqrt{2}$
all long $= l$



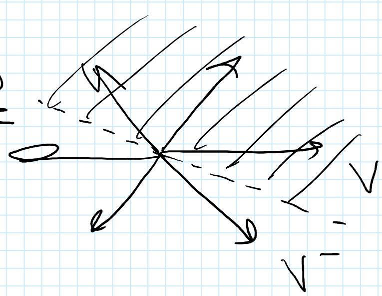
all long $= l$
all short $= \frac{l}{\sqrt{3}}$

G_2

Thm These are all root systems in rank 2.

Weyl groups = dihedral groups $\underset{\mathbb{Z}_2\times\mathbb{Z}_2}{\overset{\shortparallel}{D_2}}$, $D_3 = S_3$, D_4, D_6.

Simple roots



Pick H = hyperplane not containing any of the roots. It breaks the space into two subspaces V^+, V^-

A root is positive if it is contained in V^+

into two subspaces V^+, V^-

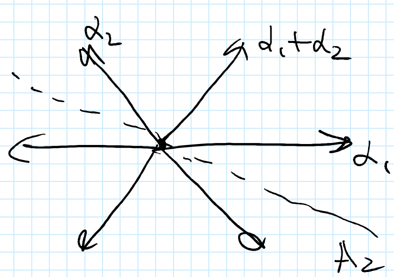
A root is <u>positive</u> if it is contained in V^+

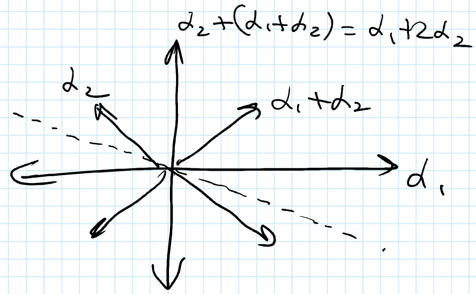
Out of $\{\alpha, -\alpha\}$ exactly one is positive.

This depends on the choice of H!

<u>Fact</u> There is a collection of <u>simple</u> positive roots $\alpha_1 \ldots \alpha_n$ in V^+ such that

- $\alpha_1 \ldots \alpha_n$ is a basis in $\mathbb{R}^n$
- Any other positive root in V^+ is a linear combination of α_i with nonnegative integer coefficients!





- Reflections in simple roots α_i generate Weyl group