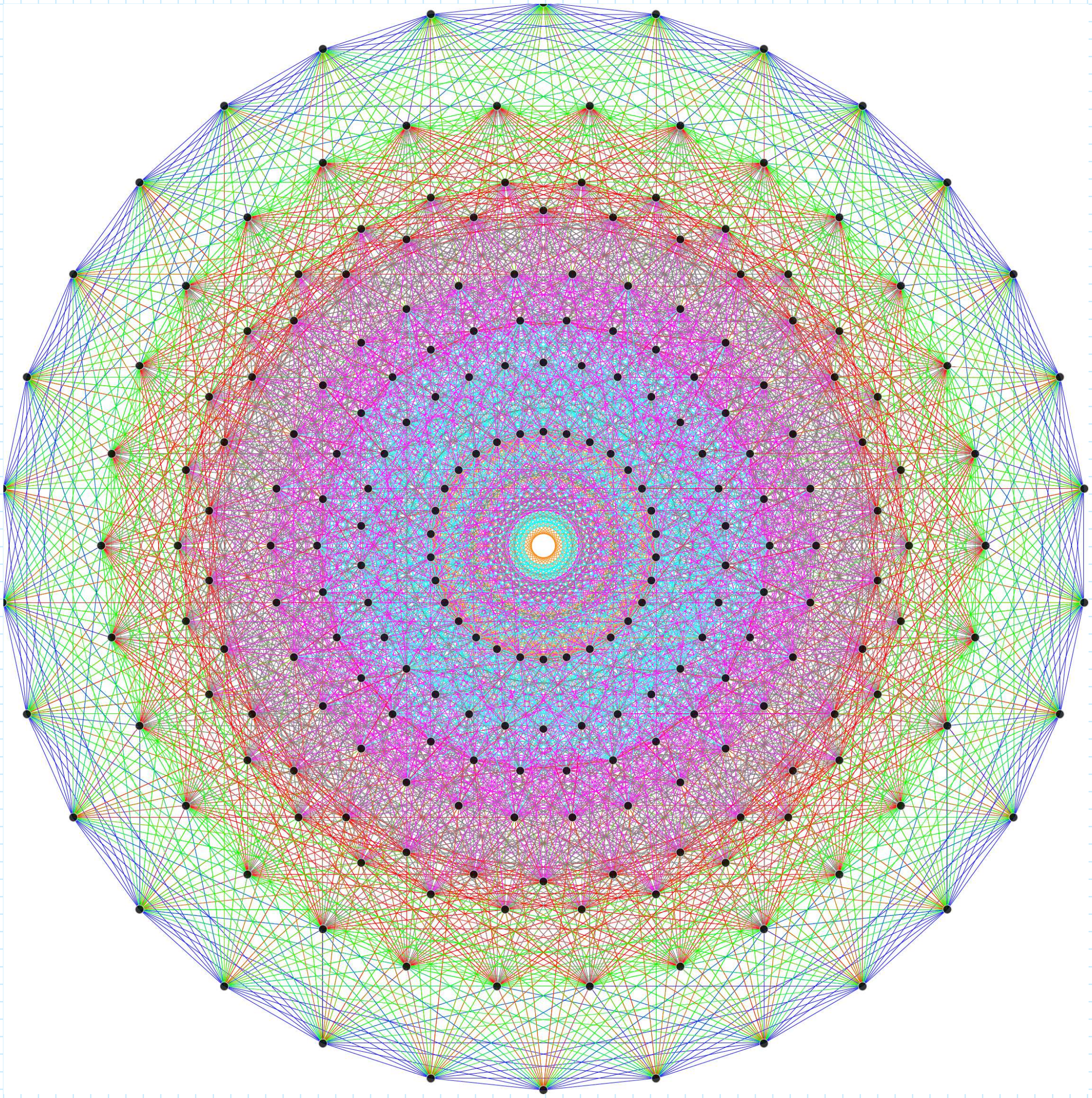
Lecture 22 (5/20)

Friday, May 20, 2022 12:12 PM

What is this?



E_8 = root system in $\mathbb{R}^8$, projected to $\mathbb{R}^2$
240 roots

<u>Note:</u> there's an element in Weyl group of order 30 which rotates this picture.

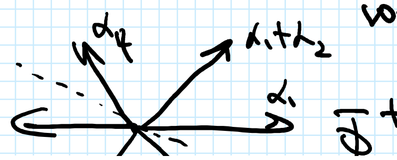
Φ = root system in $\mathbb{R}^n$

Choose a hyperplane H which does not contain any roots

$\Phi = \Phi^+ \sqcup \Phi^-$

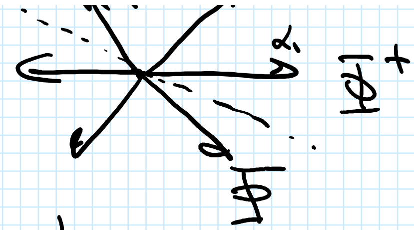
negative

positive



$\Phi = \Phi^+ \sqcup \Phi^-$

Φ^+ – positive roots; Φ^- – negative roots



Fact There exist simple roots $\alpha_1 \dots \alpha_n$ in Φ^+ such that:

- $\alpha_1 \dots \alpha_n$ is a basis in $\mathbb{R}^n$
- Any positive root is an integer nonnegative combination of α_i
- Weyl group is generated by s_{α_i}

Idea: Simple roots contain all the information about the root system Φ, we can use Weyl group to generate all roots.

Ex $\mathfrak{sl}(n) = A_{n-1}$ root system

all roots are $\pm(0 \dots 0 \underset{i}{1} 0 \dots 0 \underset{j}{-1} 0 \dots 0) = \alpha_{ij}$

simple roots

$$\left.\begin{array}{l} (1,-1,0,0,\dots\dots 0) = \alpha_1 \\ (0,1,-1,0,\dots\dots 0) = \alpha_2 \\ \vdots \\ (0 \dots\dots 0\ 1,-1) = \alpha_{n-1} \end{array}\right\} \text{$n-1$ simple roots.}$$

$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$

$\dots 1 -1 \dots$

$\dots 1 -1 \dots$

$\dots 1 -1 \dots$

Reflection s_{α_i} = simple transposition $(i\ i+1)$

Rmk Choices of Φ^+ and simple roots depend on the choice of hyperplane H, but any two choices are related by the action of Weyl group

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① Cartan matrix

$\mathbb{Z} \ni m_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ for all simple roots α_i, α_j ← $n \times n$ matrix for rank n root system

$i = j \Rightarrow m_{ii} = 2$

$n \times n$ matrix with 2's on diagonal

Lemma $i \neq j$ then $(\alpha_i, \alpha_j) \leq 0$ and $m_{ij} \leq 0$

Proof Look at rank 2 system generated by α_i, α_j and prove that the angle between them is obtuse.

Ex A_2 $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad G_2$: $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

note: $\dfrac{m_{ij}}{m_{ji}} = \dfrac{(\alpha_j, \alpha_j)}{(\alpha_i, \alpha_i)}$

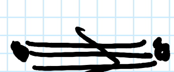
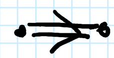
② Dynkin diagram

$B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

Graph, vertices ⟷ simple roots

edges $= (-m_{ij})$ ← two vertices i and j are connected by $(-m_{ij})$ edges.

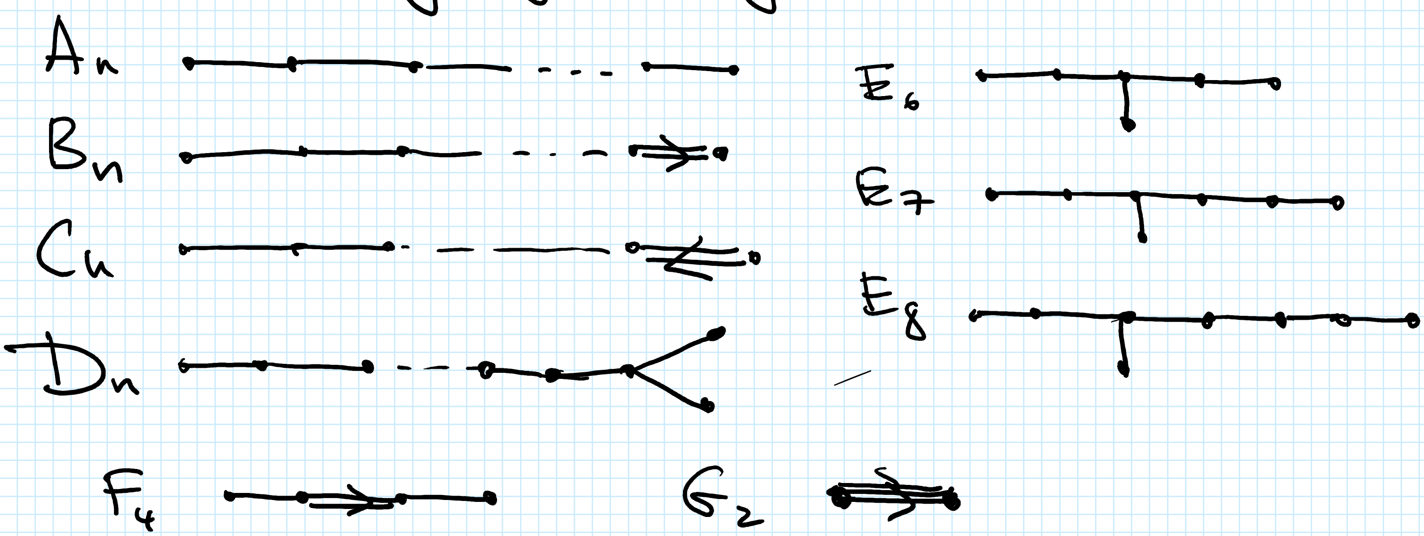
orient from longer to shorter root

A_2  $\quad G_2$  $\quad B_2$ 

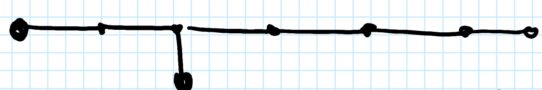
Classification Φ decomposable if $\Phi = \Phi_1 \sqcup \Phi_2$
(root system) $\qquad \Phi_1 \perp \Phi_2$

In terms of Dynkin diagram, we get a disjoint union of diagrams for Φ_1, Φ_2.

Thm All indecomposable root systems correspond the following Dynkin diagrams:



Ex. E_8



8 roots $\alpha_1 \dots \alpha_8$ in $\mathbb{R}^8$, all have same length ℓ (since simple edges only)

angles $= \frac{2\pi}{3}$ or $\frac{\pi}{2}$

if there is an edge

no edge

This determines the configuration of simple roots (up to overall scaling & rotation).

$\Rightarrow$ Weyl group $\Rightarrow$ the whole root system.

Def A simply laced root system $\Leftrightarrow$ only single edges

$\Leftrightarrow$ all roots have same length: A_n, D_n, E_6, E_7, E_8

Rmk G_2, F_4, E_6, E_7, E_8 = "exceptional" root systems

$n \geq 9$, there are exactly 4 different types of root systems.

<u>Next time</u>: Precise connection to Lie algebras
+ how to reconstruct a Lie algebra from Φ

$A_n \longleftrightarrow sl(n+1)$ $\quad$ $C_n \longleftrightarrow sp(2n)$ $\quad$ "classical Lie algebras"

$B_n \longleftrightarrow so(2n+1)$ $\quad$ $D_n \longleftrightarrow so(2n)$