Φ = root system $\alpha_1 \dots \alpha_n$ = simple roots

$$m_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \leftarrow \text{Cartan matrix}$$

Thm (Serre) We can reconstruct a semisimple complex Lie algebra $\mathfrak{g}$ from a root system:

① Generators E_i, F_i, H_i, one for each simple root

② Any element of $\mathfrak{g}$ can be obtained from these using commutators: $[E_i,[F_j,[F_k,H_\ell]]]$

③ **Relations**: $[H_i,H_j]=0 \quad \forall i,j$ ← H_i span Cartan

$[E_i,F_i]=H_i \qquad [E_i,F_j]=0 \quad i\neq j$

$[H_i,E_j]=m_{ij}E_j \qquad [H_i,F_j]=-m_{ij}F_j$ ← E_j, F_j span root subspaces for α_j

(in particular, for $i=j$ we get

$[H_i,E_i]=2E_i \qquad [H_i,F_i]=-2F_i$

so (H_i,E_i,F_i) is an $\mathfrak{sl}_2$-triple).

$$\left.\begin{array}{l} \mathrm{ad}_{E_i}^{1-m_{ij}}(E_j)=0 \\ \mathrm{ad}_{F_i}^{1-m_{ij}}(F_j)=0 \end{array}\right\} \text{Serre relations.}$$

These relations define a finite dimensional semisimple Lie algebra.

Rmk If $i\neq j$ then $m_{ij}\leq 0$, since α_i and α_j make an obtuse angle.

So $1-m_{ij}\geq 1$

obtuse angle.

So $1-m_{ij} \geq 1$

$\mathfrak{sl}_3$ $H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $H_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$E_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ $[E_1, E_2] = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$

$F_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$ $F_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}$ $[F_1, F_2] = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix}$

Serre relations: $m_{12} = -1$ since

$(\alpha_1, \alpha_2) = -1$ and $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$

$$m_{12} = \frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = \frac{2 \cdot (-1)}{2} = -1.$$

Serre relation: $\mathrm{ad}_{E_1}^{1-m_{12}}(E_2) = \mathrm{ad}_{E_1}^{2}(E_2) = [E_1, [E_1, E_2]] = 0$

$[E_2, [E_2, E_1]] = 0$

Similarly, $[F_1, [F_1, F_2]] = 0$

$[F_2, [F_2, F_1]] = 0$.

Remarks: (1) G = compact real Lie group

$\mathfrak{g} = \mathrm{Lie}(G)$ = real Lie algebra

$\mathfrak{g} \otimes \mathbb{C}$ = complex Lie algebra.

Fact This is semisimple!

Sketch: G = compact $\Rightarrow$ there's a symmetric positive definite bilinear form on any (real) representat.

$\Rightarrow$ positive definite form on $\mathfrak{g}$ $\Rightarrow$ nondegenerate form on $\mathfrak{g} \otimes \mathbb{C}$

agrees w. Killing form.

agrees w. <u>Killing form</u>. on $\mathfrak{g}\otimes\mathbb{C}$

② Conversely, a complex semisimple Lie algebra corresponds to some (possibly several) compact real Lie groups.

③ There are <u>lots</u> of important Lie algebras which are <u>not semisimple!</u>

$$\mathfrak{b}=\left\{\begin{matrix}\text{all upper}\\ \text{triangular}\\ \text{matrices}\end{matrix}\right\}\subset \mathfrak{sl}(n) \quad \text{is } \underline{\text{NOT}} \text{ semisimple}$$

More generally, if $\mathfrak{g}$ is semisimple, we can consider the <u>Borel subalgebra</u> generated by H_i and E_i (but no F_i).

$$\mathfrak{b}=\mathfrak{h}\oplus\bigoplus_{\alpha \text{ positive root}}\mathfrak{g}_\alpha \quad \leftarrow \text{equivalent definition.}$$

<u>Example</u> $so(2n)$ $2n\times 2n$ matrices, $X=-X^T$

<u>Note</u> $so(2n;\mathbb{C})$ is a complexification of $so(2n;\mathbb{R})$.

How to find Cartan subalgebra?

Divide a matrix by 2×2 blocks

$$H_1=\left(\begin{array}{cc|c|c} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & & \\ \hline & & & \end{array}\right) \qquad H_2=\left(\begin{array}{c|cc|c} & & & \\ \hline & 0 & 1 & \\ & -1 & 0 & \\ \hline & & & \end{array}\right)$$

rest $=0$ and so on.

$H_1, H_2, \dots H_n=$ generators of the Cartan, easy to see that $[H_i,H_j]=0$ for all i,j

Define matrices

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Define $E_{jk}^{(1)}, E_{jk}^{(2)}, E_{jk}^{(3)}, E_{jk}^{(4)}$ by placing $C_1 \dots C_4$ at 2×2 block at position (j,k) $j<k$ and $-C_1^T, \dots, -C_4^T$ at the block (k,j).

Ex

$$E_{12}^{(1)} = \left(\begin{array}{cc|cc} & & 1 & i \\ & & i & -1 \\ \hline -1 & -i & & \\ -i & 1 & & \end{array}\right)$$

Ex

$$\left[\overset{H_1}{\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & \end{array}\right)}, \left(\begin{array}{cc|cc} & & 1 & i \\ & & i & -1 \\ \hline -1 & -i & & \\ -i & 1 & & \end{array}\right) \right] =$$

$$= \left(\begin{array}{cc|cc} & & i & -1 \\ & & -1 & -i \\ \hline -i & 1 & & \\ 1 & i & & \end{array}\right) = i \left(\begin{array}{cc|cc} & & 1 & i \\ & & i & -1 \\ \hline -1 & -i & & \\ -i & 1 & & \end{array}\right)$$

So $[H_1, E_{12}^{(1)}] = i\, E_{12}^{(1)}$.

eigenvectors

Fact The eigenvalues of $a_1 H_1 + \dots + a_n H_n$ on $E_{jk}^{(1)} \dots E_{jk}^{(4)}$ are $i(a_j + a_k)$, $-i(a_j+a_k)$, $i(a_j - a_k)$ and $-i(a_j - a_k)$.

Conclusion Up to scaling, the roots for $so(2n)$ are $\pm(e_j + e_k)$, $\pm(e_j - e_k)$ in $\mathbb{R}^n$ where e_j is the standard basis.

Simple roots: $\underset{\alpha_1}{\underset{\shortparallel}{e_1 - e_2}}, e_2 - e_3, \dots \underset{\alpha_{n-1}}{\underset{\shortparallel}{e_{n-1} - e_n}}, \alpha_n = e_{n-1} + e_n$

Any positive root is a sum of these:

$$e_j - e_k = (e_j - e_{j+1}) + (e_{j+1} - e_{j+2}) + \dots + (e_{k-1} - e_k)$$
$$= \alpha_j + \dots + \alpha_{k-1}$$

$$e_j + e_n = (e_j - e_{n-1}) + (e_{n-1} + e_n)$$

$$e_j + e_k = (e_j + e_n) + (e_k - e_n)$$

<u>Dynkin diagram:</u> $(\alpha_i, \alpha_{i+1}) = -1$ if $i \leq n-2$ $\qquad \alpha_{n-2} = e_{n-2} - e_{n-1}$

$(\alpha_{n-2}, \alpha_n) = -1$

$(\alpha_{n-1}, \alpha_n) = 0$

$(\alpha_i, \alpha_i) = 2$ for all i.

D_n