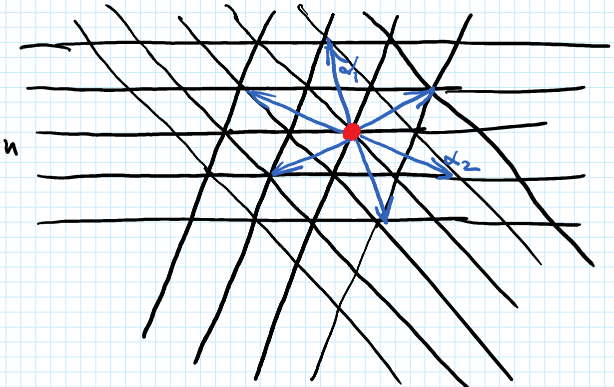
Lecture 24 (5/25)

Tuesday, May 24, 2022 11:12 PM

Representation theory!



$\mathfrak{g}$ = semisimple Lie algebra

V = finite-dimensional representation of $\mathfrak{g}$

① $X \in \mathfrak{h}$, then X diagonalizes in V

Proof Sufficient to check for H_α corresponding to roots

H_α is a part of $\mathfrak{sl}_2$-triple $(E_\alpha, F_\alpha, H_\alpha)$

And we know that in any finite-dim representation of $\mathfrak{sl}_2$ H diagonalizes!

② Since all elements of Cartan commute, they diagonalize simultaneously.

$$V = \bigoplus_\lambda V_\lambda$$

V_λ = common eigenspace for Cartan = weight space

$\lambda \in \mathfrak{h}$ such that $\boxed{Xv = (X,\lambda)v}$ for all $X \in \mathfrak{h}$ and $v \in V_\lambda$.

v = eigenvector for X

eigenvalue $= (X, \lambda)$.

$\boxed{\lambda = \text{weight if } V_\lambda \neq 0}$

③ A weight λ is called integral if $\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all roots α.

Fact V finite-dim representation $\Rightarrow$ all weights are integral

Proof α root $\Rightarrow H_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$ — part of $\mathfrak{sl}_2$-triple

In any finite-dim representation of $\mathfrak{sl}_2$, all eigenvalues, of H are integers!

$$(H_\alpha, \lambda) = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} = \text{eigenvalue of } H_\alpha \text{ on } V_\lambda.$$

Observation $\mathfrak{g}$ acts on itself by adjoint representation

weights for adjoint = roots of $\mathfrak{g}$.

Pictures: how to draw integral weights?

Integral weights form a lattice in $\mathfrak{h}$:

$\alpha_1, \dots, \alpha_n$ = simple roots

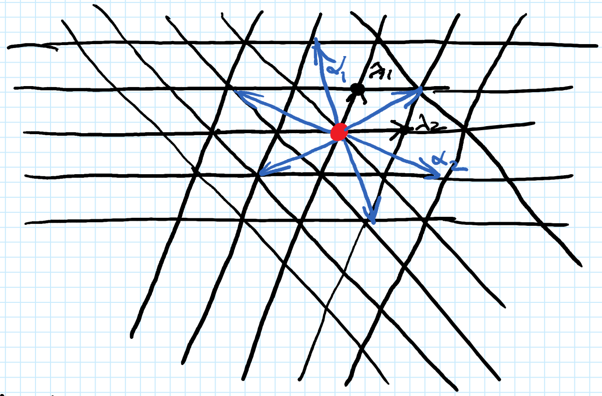
$\lambda_1 \dots \lambda_n$ are defined by $\frac{2(\alpha_i, \lambda_i)}{(\alpha_i,\alpha_i)} = 1$ $\quad \frac{2(\alpha_i,\lambda_j)}{(\alpha_i,\alpha_i)} = 0$ $\quad i \neq j$

$\approx$ dual basis

Fact λ = integral weight $\iff \lambda = m_1\lambda_1 + \dots + m_n\lambda_n$ where $m_i \in \mathbb{Z}$.

Λ = lattice generated by λ_i

A_2



$(\alpha_1,\alpha_1) = 2 \qquad (\alpha_1,\alpha_2) = -1$

$(\alpha_2,\alpha_2) = 2$

$$\frac{2(\alpha_1,\lambda_1)}{(\alpha_1,\alpha_1)} = (\alpha_1,\lambda) = 1$$

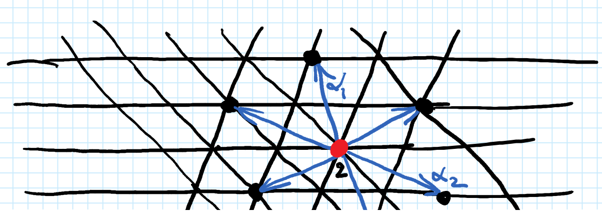
$(\alpha_2,\lambda_1) = 0 \qquad \lambda_1 \perp \alpha_2$

$\lambda_2 \perp \alpha_1$

$(\lambda_2,\alpha_2) = 1.$ $\qquad \lambda$ = any integral weight $\Rightarrow \lambda = m_1\lambda_1 + m_2\lambda_2$

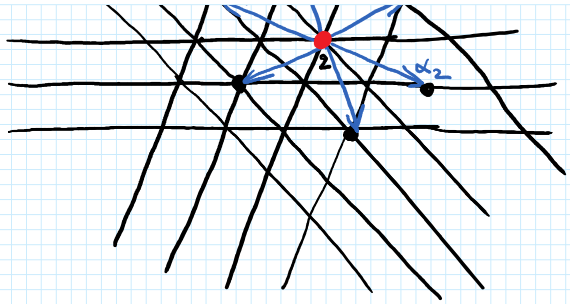
V = any f.d. representation of sl_3 $\longrightarrow V = \bigoplus V_\lambda \quad \lambda \in \Lambda$

Ex Adjoint representation of sl_3



$Ad = V_{\alpha_1} \oplus V_{\alpha_2} \oplus V_{\alpha_1+\alpha_2} \oplus$

$\oplus V_{-\alpha_1} \oplus V_{-\alpha_2} \oplus V_{-\alpha_1-\alpha_2} \oplus$



$\oplus V_{-\alpha_1} \oplus V_{-\alpha_2} \oplus V_{-\alpha_1-\alpha_2} \oplus$

$\oplus V_0$, $\dim V_0 = 2$

$V_0 \simeq \mathfrak{h}$ Cartan.

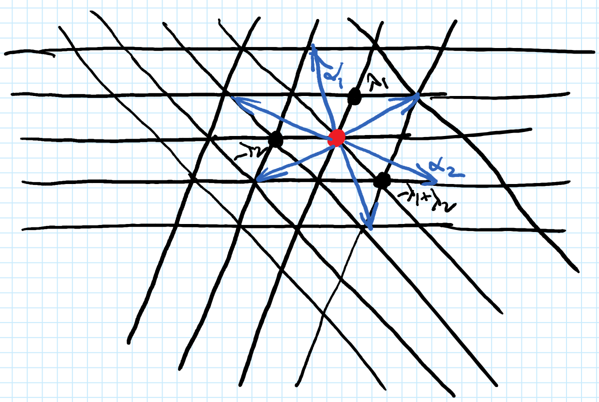
General question: given V, which weights appear and $\dim V_\lambda = ?$

<u>Ex</u>: sl_3 acts on $\mathbb{C}^3$! Fundamental/defining representation.

$$\alpha_1 \longleftrightarrow \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = H_1 \qquad \alpha_2 \longleftrightarrow \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} = H_2$$

$e_1, e_2, e_3 =$ standard basis in $\mathbb{C}^3$

$H_1 e_1 = e_1$	$H_1 e_2 = -e_2$	$H_1 e_3 = 0$
$H_2 e_1 = 0$	$H_2 e_2 = e_2$	$H_2 e_3 = -e_3$
weight λ_1	weight $-\lambda_1+\lambda_2 = \lambda$	weight $-\lambda_2 = 0\cdot\lambda_1 - \lambda_2$
$(\alpha_1, \lambda_1) = 1$	$(\alpha_1, \lambda) = -1$	$(\alpha_1, -\lambda_2) = 0$
$(\alpha_2, \lambda_1) = 0$	$(\alpha_2, \lambda) = 1$	$(\alpha_2, -\lambda_2) = -1$



$X \in \mathfrak{h}$ acts on V_λ with eigenvalue (X, λ)

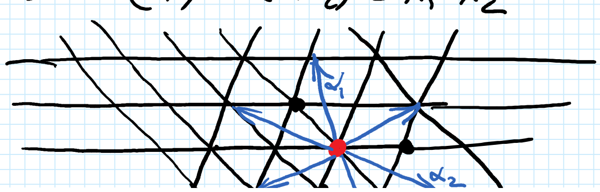
$\Lambda^2 \mathbb{C}^3$, basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$

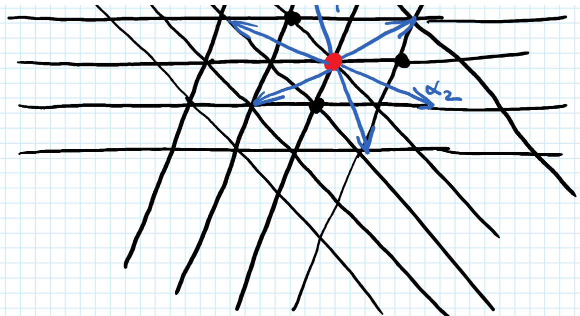
$\parallel$

$(\mathbb{C}^3)^*$ eigenvalues for $\otimes$, $\wedge$, Sym just add up.

$e_1 \wedge e_2 \longrightarrow \underset{e_1}{(\lambda_1)} + \underset{e_2}{(-\lambda_1 + \lambda_2)} = \lambda_2$ $\qquad$ $e_2 \wedge e_3 \rightarrow (-\lambda_1 + \lambda_2) + (-\lambda_2) =$

$e_1 \wedge e_3 \longrightarrow (\lambda_1) + (-\lambda_2) = \lambda_1 - \lambda_2$ $\qquad\qquad\qquad$ $= -\lambda_1$

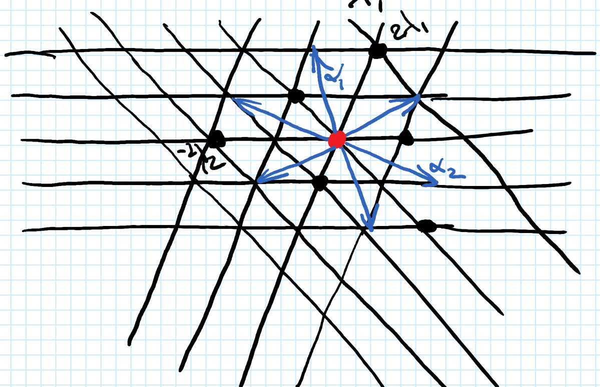




Fact $V = \bigoplus V_\lambda$ then $V^* = \bigoplus (V_\lambda)^* = \bigoplus V_{-\lambda}$

weight on $(V_\lambda)^*$ equals $-\lambda$.

$\mathrm{Sym}^2 \mathbb{C}^3$: basis
$e_1^2 \longrightarrow \lambda_1 + \lambda_1 = 2\lambda_1$
$e_2^2 \longrightarrow 2(-\lambda_1 + \lambda_2)$
$e_3^2 \longrightarrow -2\lambda_2$
$\left.\begin{array}{l} e_1 e_2 \longrightarrow \lambda_2 \\ e_1 e_3 \longrightarrow \lambda_1 - \lambda_2 \\ e_2 e_3 \longrightarrow -\lambda_1 \end{array}\right]$ as above.



Observation 1: The diagram is symmetric! The Weyl group $= S_3 = D_3$ acts on the weight diagram.

This is true in general, proof next time.

Observation 2: For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$, we can actually consider $\lambda \in \mathbb{C}^n$ any diagonal matrix.

λ defined up to a multiple of $(1, \ldots, 1)$

Start from $\lambda \in \mathbb{C}^n$, then project to Cartan.

In this way, in $V = \mathbb{C}^3$ we can choose the weight for $e_1 = (1,0,0) \xrightarrow{\text{project}} \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) = \lambda_1$

for $e_1 = (1,0,0) \xrightarrow[\text{to Cartan}]{\text{project}} \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) = \lambda_1$