Lecture 25 (5/27)

Friday, May 27, 2022 1:06 PM

Recap: V = representation of $\mathfrak{g}$

$V = \bigoplus V_\lambda$ $\quad \lambda$ - weights

$X \in \mathfrak{h}$ acts on V_λ with eigenvalue (X, λ)

V_λ = common eigenspace for Cartan.

Recall for simple roots $\alpha_1 \dots \alpha_n$ we choose $E_1 \dots E_n$, $F_1 \dots F_n$

Lemma $E_i V_\lambda \subset V_{\lambda + \alpha_i}$

$F_i V_\lambda \subset V_{\lambda - \alpha_i}$

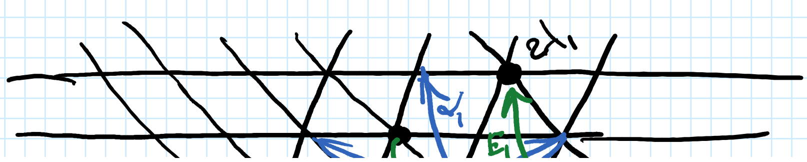
Proof Suppose $v \in V_\lambda$ and $X \in \mathfrak{h}$, then we need to show $E_i v \in V_{\lambda + \alpha_i}$. Observe:

$$X(E_i v) = E_i (X v) + [X, E_i] v =$$

$$= E_i \underbrace{(X, \lambda)}_{v \text{ is in } \lambda\text{-weight subspa}} v + \underbrace{(X, \alpha_i)}_{E_i \text{ is in } \alpha_i\text{-root subspace in } \mathfrak{g}!} E_i v =$$

$$= (X, \lambda) \cdot E_i v + (X, \alpha_i) E_i v =$$

$$= (X, \lambda + \alpha_i) E_i v \Rightarrow E_i v \in V_{\lambda + \alpha_i} \quad \blacksquare$$



Weight diagram

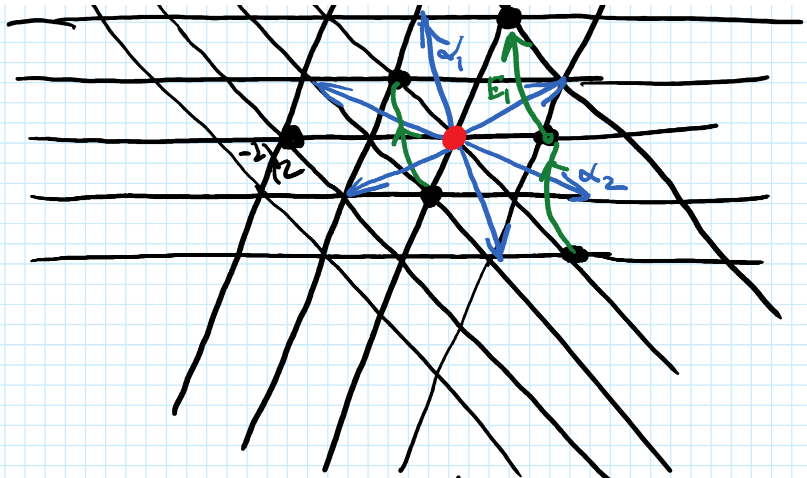


diagram for $\mathrm{Sym}^2\mathbb{C}^3$ as representation of $\mathfrak{sl}_3$.

E_1 adds α_1 to the weight, F_1 subtracts α_1

E_2 adds α_2 to the weight, F_2 subtracts α_2

<u>Ex</u>: $\mathfrak{sl}_2(E_1, F_1, H_1)$ acts on $\mathrm{Sym}^2\mathbb{C}^2 = V$

as $\mathfrak{sl}_2$-representation, V breaks into $L(2)+L(1)+L(0)$

<u>Observation</u>: Using the weight diagram and Lemma above, we have a very good control on the action of E_i, F_i in V.

<u>Remark</u> For this, we need to choose a basis in each V_λ, this could be hard!

<u>Thm</u> The weight diagram is preserved by the action of Weyl group

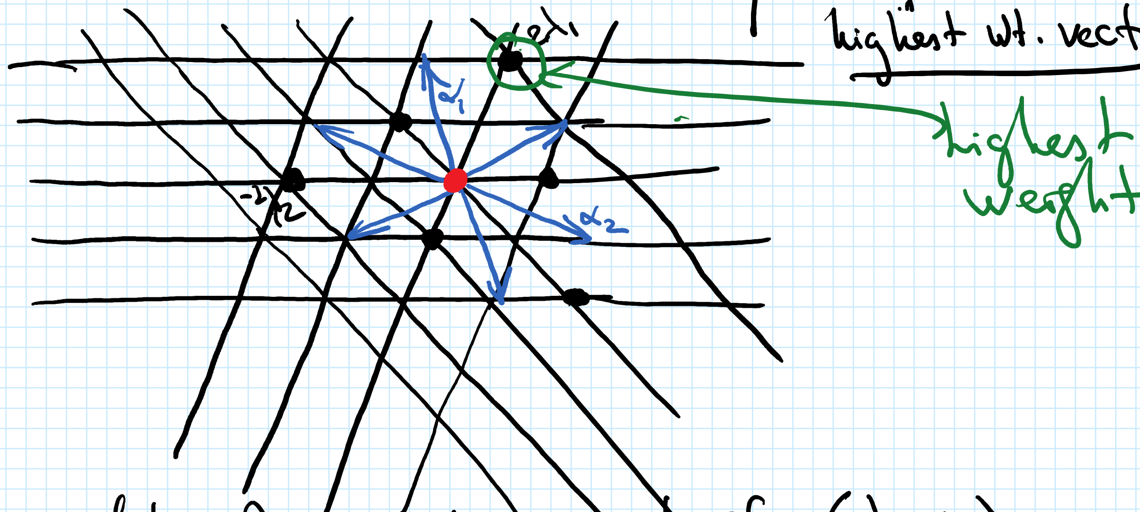
<u>Proof</u>: Weyl group is generated by reflections s_i in hyperplanes $\perp \alpha_i$.

Choose $\mathfrak{sl}_2(H_i, F_i, E_i)$, as $\mathfrak{sl}_2$-representation

V is symmetric, so weight diagram breaks into symmetric chains $\Rightarrow$ symmetric wrt S_i. $\blacksquare$

Def v is a highest weight vector in V if
- v is an eigenvector for all H_i
- $E_i v = 0$ for all i.

Highest weight = weight of a highest wt. vector



Def A weight λ is dominant if $(\lambda, \alpha_i) \geq 0$ for all α_i.

Lemma A highest weight is always dominant.

Proof: Restrict to $sl_2(E_i, F_i, H_i)$. Let v be a highest wt vector. H_i has eigenvalue $\frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ on V

$E_i v = 0 \Rightarrow v$ is highest weight for $sl_2 \Rightarrow$

$\Rightarrow \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ is a nonnegative integer! $\blacksquare$

Thm (Highest weight theorem) V = finite dim representation of g

(a) V has a highest weight vector

(b) V is irreducible if and only if it has a unique highest weight vector (up to scaling)

(c) If V is irreducible with highest weight μ then all other weights have the form

$$\mu - \sum k_i \alpha_i \quad \text{where } k_i \in \mathbb{Z}_{\geq 0}$$

(d) If V irreducible, v is the highest wt. vector then V is generated by v under the action of F_i.

(e) For any dominant integral weight μ there is a unique irreducible finite-dim. representation of $\mathfrak{g}$ with highest weight μ.

Thm $\mathfrak{g}$ = semisimple Lie algebra $\Rightarrow$ any representation of $\mathfrak{g}$ is a direct sum of irreducibles.

What does it mean in practice?

① $\mathfrak{g} = \mathfrak{sl}_n \quad \alpha_i = (0 \ldots 1, -1, 0 - - 0)$

$\lambda = (\lambda_1, - \cdot - \lambda_n)$

$(\lambda, \alpha_i) = \lambda_i - \lambda_{i+1}$

λ dominant $\Leftrightarrow \lambda_i - \lambda_{i+1} \geq 0$ for all i

$\Leftrightarrow \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq - - \geq \lambda_n$

λ integral $\Leftrightarrow \lambda_i - \lambda_{i+1} \in \mathbb{Z}$

Observation: we can add a multiple of $(1 - - 1)$ to λ

without changing its projection to the Cartan.

Ways to fix it: (1) Set $\lambda_n = 0$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$$

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z} \Rightarrow \lambda_i \in \mathbb{Z}_{\geq 0}$$

Such λ is a partition with at most $(n-1)$ parts.

Dominant integral weights for sl_n $\longleftrightarrow$ (partitions with at most $(n-1)$ parts)

$\updownarrow$ by Thm

irreducible representations of $sl(n)$.

Example/exercise: $\mathbb{C}^3 \longrightarrow (1,0,0)$

$\mathrm{Sym}^2\mathbb{C}^3 \longrightarrow (2,0,0)$

$\Lambda^2\mathbb{C}^3 \longrightarrow (1,1,0)$

(2) $\lambda \in \mathrm{Cartan}(sl(n)) \Leftrightarrow \sum \lambda_i = 0$

$$(\lambda_1, \dots \lambda_n) \longrightarrow (\lambda_1 - \bar{\lambda}, \lambda_2 - \bar{\lambda}, \dots \lambda_n - \bar{\lambda})$$

$$\bar{\lambda} = \frac{\lambda_1 + \dots + \lambda_n}{n}$$

$$(1,0,0) \longrightarrow \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$