Lecture 26 (6/1)

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Characters

Def $f(x_1 \ldots x_n)$ is called symmetric if

$$f(x_1 \ldots x_{i+1}, x_i, \ldots x_n) = f(x_1 \ldots x_i, x_{i+1} \ldots x_n)$$

antisymmetric (or alternating) if

$$f(x_1 \ldots x_{i+1}, x_i, \ldots x_n) = -f(x_1 \ldots x_i, x_{i+1}, \ldots x_n)$$

Symmetric, if it is preserved by simple transpositions $s_i = (i \; i+1)$

$\Leftrightarrow$ preserved by any permutation in S_n

Antisymmetric, if $s_i f = -f$, equivalently,

$$w \cdot f = \operatorname{sgn}(w) \cdot f \quad \text{for any } w \in S_n$$

Lemma f is antisymmetric polynomial $\Leftrightarrow$ f is divisible by $\prod_{i<j}(x_i - x_j) = \Delta$ (antisymmetric)

and $\frac{f(x)}{\Delta}$ is a symmetric polynomial.

Proof: f antisymmetric $\Rightarrow f(x_1 \ldots x_i \ldots x_j \ldots x_n) =$

$$= -f(x_1 \ldots x_j \ldots x_i \ldots x_n)$$

transposition $(i \; j)$

$\operatorname{sgn}(i \; j) = -1$

Plug in $x_i = x_j$, get

$$f(x_1 \ldots x_i \ldots x_i \ldots x_n) = -f(x_1 \ldots x_i \ldots x_i \ldots x_n)$$

$\Rightarrow f(x) = 0$ for $x_i = x_j$ $\Rightarrow$ $f(x)$ is divisible by $x_i - x_j$

hyperplane! for all $i \neq j$

$\mathbb{C}[x_1 \ldots x_n]$ is a UFD, $x_i - x_j$ all coprime

$\Rightarrow f(x)$ is divisible by $\prod_{i<j}(x_i - x_j) = \Delta$. ∎

Lemma (Vandermonde det.)

$$\begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = \pm \Delta = \pm \prod_{i<j}(x_i - x_j)$$

Proof This determinant is antisymmetric in $x_i \Rightarrow$ divisible by Δ, degree $\left(\text{as polynomial in } x_i\right) = \binom{n}{2} = \deg \Delta$

$\Rightarrow \frac{\det}{\Delta}$ is a constant. ∎

All these facts can be generalized to any root system!

W = Weyl group acts on $\mathfrak{h}$ = Cartan subalgebra

f = polynomial function on Cartan

Def f is W-invariant, if $w \cdot f = f$ for all $w \in W$

$\Leftrightarrow s_\alpha f = f$ for all roots α

f is W-antiinvariant if $w \cdot f = \mathrm{sgn}(w) \cdot f$ for $w \in W$

$\Leftrightarrow s_\alpha f = -f$ for all roots α.

Lemma f is W-antiinvariant $\Leftrightarrow f$ is divisible by

$\Delta = \prod_{\alpha \in \Phi^+} (x, \alpha) \Leftarrow$ product of linear functn of $x \in \mathfrak{h}$

positive roots

Generalizing $x_i - x_j$ in type A

Example $\alpha = (0 - 1 - \cdots - 1 - 0)$

and f/Δ is W-invariant.

roots

and f/Δ is W-invariant.

Also, Δ is W-antiinvariant.

Example

$\alpha = (0 \text{—} 1 \text{——} -1 \text{——} 0)$

$X = (x_1 \text{--} \cdots \text{--} x_n)$

then $(x, \alpha) = x_i - x_j$

Proof: exercise, same as type A.

Rmk: $\det A = \sum_{w \in S_n} (\cdot \text{——}) \cdot \operatorname{sgn}(w)$

We want to generalize it by $\sum_{w \in W} (\cdots) \operatorname{sgn}(w)$.

Lemma (generalized Vandermonde)

$$\sum_{w \in W} \operatorname{sgn}(w) \cdot x^{w(\rho)} = \pm \Delta = \pm \prod_{\alpha \in \Phi^+} (x, \alpha)$$

Here ρ = half sum of all positive roots $= \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

and $x^\alpha = \prod x_i^{\alpha_i}$ for some choice of basis in Cartan.

Proof: LHS is antisymmetric in x — not hard

By previous lemma, it is divisible by Δ

By comparing the degrees, we see that $\frac{f}{\Delta} = $ const. ∎

Rmk In type A, $\rho = (n-1, n-2, \ldots 0)$ ← exercise.

Def Schur polynomial $\lambda = (\lambda_1 \ldots \lambda_n)$ $\lambda_1 \geq \cdots \geq \lambda_n$

$$S_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\Delta} = \frac{\det(x_i^{\lambda_j + n - j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n}$$

Ex $n=3$, $\lambda=(1,1,0)$ $\rho=(2,1,0)=(n-1,n-2,\dots,0)$

$$(\lambda_j+n-j)=\lambda+\rho=(3,2,0)$$

$$S_{1,1,0}=\frac{\begin{vmatrix} x_1^3 & x_1^2 & x_1^0 \\ x_2^3 & x_2^2 & x_2^0 \\ x_3^3 & x_3^2 & x_3^0 \end{vmatrix} \leftarrow x_i^{\lambda_j+n-j}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} \leftarrow x_i^{n-j}}$$

Lemma S_λ is a symmetric polynomial in x_i !

Proof Top is antisymmetric polynomial, bottom $=\Delta$.

Rmk S_λ is homogeneous, $\deg=|\lambda|=\sum\lambda_i$
(compare degrees on top & bottom).

Thm (Weyl character formula)

① Define a "generalized Schur polynomial"

$$S_\lambda=\frac{\sum_{w\in W} \operatorname{sgn}(w)\cdot x^{w(\lambda+\rho)}}{\sum_{w\in W} \operatorname{sgn}(w)\cdot x^{w(\rho)}} \leftarrow \Delta!$$

Defined for $\lambda=$ dominant integral weight in Cartan
(want all powers to be integers to talk about polynomials)

② S_λ is a W-invariant polynomial on Cartan

(top = W-"antisymmetric" function)

③ S_λ form a basis of all W-invariant polynomials!

④ $V(\lambda)$ = irreducible representation of $\mathfrak{g}$ with highest weight λ

$$V(\lambda) = \bigoplus_\mu V(\lambda)_\mu$$ — weight subspaces in $V(\lambda)$ of weight μ.

$$\boxed{\operatorname{ch} V(\lambda) := \sum_\mu x^\mu \dim V(\lambda)_\mu = \text{character of } V(\lambda)}$$

⑤ Weyl character formula:

$$\boxed{\operatorname{ch} V(\lambda) = S_\lambda}$$