Lecture 3 (4/1)

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The unitary group

$\mathbb{C}^n$, Hermitian product

$$\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$$

A is unitary if it preserves $\langle , \rangle$:

$$\boxed{\langle Ax, Ay\rangle = \langle x, y\rangle \text{ frall } x, y.}$$

$$\langle x, y\rangle = \overline{x}^T y$$

$$\langle Ax, Ay\rangle = (\overline{Ax})^T (Ay) = \overline{x}^T \overline{A}^T A y$$

$$= x^* A^* A y \qquad A^* = \overline{A}^T$$

This equals $\langle x, y\rangle$ if $\boxed{A^* A = I}$

Facts ① A unitary $\iff A^* A = I$

$\iff$ columns of A form an orthonormal basis wrt $\langle , \rangle$

$$\langle x, x\rangle = 1 \iff \sum \overline{x_i} x_i = 1 = \sum |x_i|^2$$

② A unitary $\Rightarrow$ A invertible $A^{-1} = A^*$ also unitary

A, B unitary $\Rightarrow AB$ unitary

Def $U(n)$ = unitary group = { unitary matrices }

$SU(n)$ = special unitary group = $\{A : A \text{ unitary}, \det A \overset{!}{=} 1\}$

③ A unitary $\Rightarrow |\det A| = 1$

$$\det(A^* A) = \overline{\det(A)} \cdot \det A = 1$$

Ex $U(1) = \{a : \bar{a}a = 1 = |a|\}$ = unit circle in $\mathbb{C}$
$\quad \| \; S^1 \qquad SU(1) = \{1\}$

Ex $SU(2) = \left\{ \begin{pmatrix} \alpha & x \\ \beta & y \end{pmatrix} : \begin{array}{l} \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \\ \bar{\alpha}x + \bar{\beta}y = 0 \\ \bar{x}x + \bar{y}y = 1 \\ \det = 1 \end{array} \right\}$

$\begin{pmatrix} x \\ y \end{pmatrix} = k \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}$

$\begin{pmatrix} \alpha & \\ \beta & \end{pmatrix}$

Conclusion $SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \right.$

And $SU(2) \simeq S^3$!

Facts ① $U(n)$ and $SU(n)$ are compact.

Closed since $A^*A = I$, bounded since $|a_{ij}|^2 \leq 1$
this continuous

② $U(n)$ and $SU(n)$ are connected

Proof: same as $O(n)$

③ $GL(n, \mathbb{C})$ retracts onto $U(n)$

Proof: same as for $O(n)$ last time

Gram-Schmidt process.

Note: $O(n) \subset U(n)$ $\qquad A$ real $\Rightarrow A^* = A^T$

$SO(n) \subset SU(n)$

$SO(2) = S^1 \qquad SU(2) = S^3$

Cor. $\cdot\, GL(n, \mathbb{C}) \sim U(n)$ connected

$\cdot\, GL(n, \mathbb{R}) \sim O(n)$ has exactly 2 connected components.

• $GL(n,\mathbb{R}) \cap O(n)$ has exactly 2 connected components.

<u>Thm</u> There is a surjective homomorphism $\varphi: SU(2) \longrightarrow SO(3)$ such that $\operatorname{Ker}\varphi = \{\pm I\}$

<u>Cor</u> $SO(3) = SU(2)/\operatorname{Ker}\varphi = S^3/\{\pm I\} = \mathbb{R}P^3$ (Exercise)

<u>Plan:</u> We want to do the following:

• Find a real 3d vectorspace $\mathcal{U}$ + symmetric positive definite bilinear form $(\,,)$

• Find an action of $SU(2)$ on $\mathcal{U}$ which preserves $(\,,)$ $\Rightarrow \varphi: SU(2) \to SO(3)$

<u>Construction</u> • $\mathcal{U} = \left\{ X \in \mathrm{Mat}(2\times 2) \;\middle|\; \begin{array}{l} X = \overline{X}^T = X^* \\ Tr(X) = 0 \end{array} \right\}$

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{array}{c} a = \overline{a} \\ b = \overline{c} \\ d = \overline{d} \\ a+d=0 \end{array} \quad = \begin{pmatrix} x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -x_1 \end{pmatrix} \quad x_1, x_2, x_3 \in \mathbb{R}$$

• $(X, Y) = \frac{1}{2} Tr(XY)$

$$XY = \begin{pmatrix} x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -x_1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 + i y_3 \\ y_2 - i y_3 & -y_1 \end{pmatrix} = \begin{bmatrix} x_1 y_1 + x_2 y_2 - i x_2 y_3 + i x_3 y_2 + x_3 y_3 & * \\ * & x_2 y_2 + i x_2 y_3 - i x_3 y_2 + x_3 y_3 + x_1 y_1 \end{bmatrix}$$

$Tr(XY) = 2(x_1 y_1 + x_2 y_2 + x_3 y_3)$ ← this is the usual dot product in $\mathbb{R}^3$

• Action of $SU(2)$:

A unitary, $X \in \mathcal{U}$ $\quad A(X) = AXA^*$

$$\left.\begin{array}{l} A \text{ unitary}, X \in \mathcal{U} \qquad A(X) = AXA^* \\ (AXA^*)^* = A^{**}X^*A^* = AX^*A^* \\ \text{if } X^* = X \text{ then } (AXA^*)^* = (AXA^*) \\ Tr(AXA^*) = Tr(AXA^{-1}) = Tr(X) = 0 \end{array}\right\} A(X) \in \mathcal{U}.$$

Action: $(AB)X(AB)^* = ABXB^*A^* = A(BXB^*)A^*$.

- Action preserves the form:

$$Tr(AXA^* \cdot AYA^*) = Tr(AXYA^*) = Tr(AXYA^{-1}) = Tr(XY).$$

This means we have homomorphism $\varphi: SU(2) \longrightarrow O(3)$.

<u>Note:</u> $SU(2)$ connected, φ is continuous $\Rightarrow$ image is connected.

$Im(\varphi)$ contains $I \Rightarrow$ in $SO(3)$.