Lecture 4 (4/6)

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$$\varphi : SU(2) \to SO(3)$$

$A \longrightarrow$ transformation in

$$\mathcal{U} = \left\{ X \in Mat(2\times 2) \,\middle|\, \begin{array}{l} X = \bar{X}^T \\ Tr(X) = 0 \end{array} \right\}.$$

$$X \longrightarrow AXA^*$$

Claim: $Ker(\varphi) = \{\pm I\}$

Proof: A is in $Ker(\varphi) \iff$ for all X in $\mathcal{U}$ $\quad AXA^* = AXA^{-1} = X$

$$AXA^{-1} = X \iff AX = XA$$

Conclusion: $Ker(\varphi) = \{A \in SU(2) : A$ commutes with all matrices in $\mathcal{U}\}$

Pick $\tilde{X}(x_1) = \begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix}$ $\quad x_1 \neq 0$

Very important fact:

$$M = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

diagonal, $a_i \neq a_j$

$MA = AM \Rightarrow A$ diagonal

$m_{ij} a_i = m_{ij} a_j \quad \forall i, j$

If $i \neq j$, $m_{ij} = 0$

If A commutes with $\tilde{X}(x_1) \Rightarrow A$ diagonal

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\lambda_1 \neq \lambda_2 \Rightarrow$ only diagonal matrices commute with A

$\Rightarrow$ contradiction.

So $\lambda_1 = \lambda_2$.

If $i \neq j$, $m_{ij} = 0$

so $\lambda_1 = \lambda_2$.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \det A = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

(we are in $SU(2)$)

Surjectivity: see the book (1.4), or postponed

Thm $\pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}_2, & n \geq 3 \end{cases}$

Proof $n=2$ $SO(2) = S^1$, so $\pi_1(S^1) = \mathbb{Z}$

$n=3$ $SO(3) = SU(2)/_{\pm 1} = S^3/_{\pm 1} = \mathbb{RP}^3$, $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$

For $n > 3$:

We have an action of $SO(n)$ on $\mathbb{R}^n$

HW: orbit of $e_1 = (1, 0 \dots 0)$ is $S^{n-1} = \{$all vectors of length 1$\}$

stab $= SO(n-1)$

$\Rightarrow$ We have a locally trivial fibration

$$\begin{aligned} SO(n) &\longrightarrow S^{n-1}, \quad \text{fiber} = SO(n-1) \\ A &\longmapsto Ae_1 \end{aligned}$$

Long exact sequence of fibration:

$$\to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F)$$

$$\to \pi_2(S^{n-1}) \to \pi_1(SO(n-1)) \to \pi_1(SO(n)) \to \pi_1(S^{n-1})$$

$n>3$, $n-1>2$: $\pi_2(S^{n-1}) = 0$, $\pi_1(SO(n-1))$ is F, $\pi_1(SO(n))$ is E, $\pi_1(S^{n-1})$ is $B = 0$.

Therefore $\pi_1(SO(n)) \simeq \pi_1(SO(n-1))$ ✱

Complexification $\quad \mathbb{C}^n \simeq \mathbb{R}^{2n}$ (as real vector spaces)

$$e_1 \dots e_n \longleftrightarrow e_1 \dots e_n, ie_1 \dots ie_n$$

$$GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$$

$M = X + iY$

$X, Y =$ real $n \times n$ matrices

$$Me_k = \sum m_{sk} e_s = \\ = \sum (x_{sk} + iy_{sk}) e_s \\ = \sum x_{sk} e_s + \sum y_{sk} \cdot (ie_s)$$

$$M \longrightarrow A = \left(\begin{array}{c|c} X & -Y \\ \hline Y & X \end{array}\right)$$

$M(ie_k) =$
$= -\sum y_{sk} e_s +$
$+ \sum x_{sk} (ie_s)$

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

$\cap$

$U(1) \longrightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$

Lemma $\quad J = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$ matrix of multiplication by i on $\mathbb{C}^n$

$$A \in \mathrm{Im}(GL(n, \mathbb{C})) \Longleftrightarrow AJ = JA.$$

Symplectic group

$\mathbb{R}^{2n}$, define a bilinear form

$\omega(x,y) = (x, Jy)$ where $(,)$ is the dot product and J as above.

This is antisymmetric in x and y:

$$\omega(x,y) = x^T J y = (x^T J y)^T = y^T J^T x = -y^T J x = -\omega(y,x)$$

since $J^T = -J$.

ω is called the symplectic form on $\mathbb{R}^{2n}$

Def The (real) symplectic group

$$Sp(n,\mathbb{R}) = \{A : \omega(Ax, Ay) = \omega(x,y)\} < GL(2n,\mathbb{R})$$

That is, A preserves the form.

Note If A, B preserve ω, then A^{-1}, AB preserve $\omega \Rightarrow Sp(n,\mathbb{R})$ is a group. (In fact, a matrix Lie group)

$\omega(x,y) = x^T J y$

$\omega(Ax, Ay) = (Ax)^T J (Ay) = x^T A^T J A y$

So $A \in Sp(n,\mathbb{R}) \iff \boxed{A^T J A = J}$

Fact $Sp(n,\mathbb{R}) \cap O(2n) \simeq U(n)$

Idea: $Sp(n,\mathbb{R}) \longrightarrow A^T J A = J$

$O(2n) \longrightarrow A^T = A^{-1}$

If we combine these, we get $A^{-1} J A = J$

or $JA = AJ$

$\Rightarrow A \in GL(n,\mathbb{C})$.