Lecture 5 (4/8)

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V = vector space over $\mathbb{K}$

char $\mathbb{K} \neq 2$

Bilinear form on V:

$m : V \otimes V \longrightarrow \mathbb{K}$ $\quad x, y \in V$

$m(x, y) \in \mathbb{K}$ linear in x and y separately

$$m(\alpha x_1 + \beta x_2, y) = \alpha m(x_1, y) + \beta m(x_2, y)$$

Fact If choose a basis in V, any bilinear form corresponds to a matrix M such that $\boxed{m(x, y) = x^T M y}$

$M = I \Longrightarrow$ usual dot product in an orthonormal basis

Fact m is symmetric $\Longleftrightarrow m(x, y) = m(y, x)$

iff $M^T = M$

m is antisymmetric $\Longleftrightarrow m(x, y) = -m(y, x)$

iff $M^T = -M$.

Group of isometries of $m(x, y)$

$= \{ A \in GL(V) : m(Ax, Ay) = m(x, y)$ for all $x, y \}$

Fact i) A is an isometry for $M \Longleftrightarrow$

$$m(Ax, Ay) = (Ax)^T M (Ay) = x^T A^T M A y$$

$\boxed{A^T M A = M}$ (*)

$m(Ax, Ay) = (Ax)^T M (Ay) = x^T A^T M A y$

$$\boxed{A^T M A = M} \quad (*)$$

Note: If $\operatorname{rank}(M) = n$, then $(*) \Rightarrow A$ invertible

$$(M^{-1} A^T M) \cdot A = I$$

$$A^{-1} = M^{-1} A^T M.$$

2) A, B are isometries for $M \Rightarrow AB$ is an isometry for M, and A^{-1} is an isometry for M

$\Rightarrow \mathrm{Iso}(M) = \{$group of all isometries of $M\}$

LHS of $(*)$ is continuous in A

$\Rightarrow \mathrm{Iso}(M)$ is closed in $GL(V)$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$)

$\Rightarrow \mathrm{Iso}(M)$ is a matrix Lie group.

Rmk $\mathbb{K}$ = finite field, $\mathrm{Iso}(M)$ is a finite group (finite group of Lie type, ex. $O(n, \mathbb{K})$).

M symmetric $\mathbb{K} = \mathbb{R}$

$m(x,y)$ = symmetric bilinear form.

Very important fact We can choose a basis in $V = \mathbb{R}^n$

where $M = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & 0 & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{pmatrix}$ $\quad$ $p \quad +1$, $m \quad -1$, $s \quad 0$

$p + m + s = n$

$$\begin{pmatrix} \ddots & & & & \\ & \underbrace{-1}_{m} & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & \underbrace{0}_{s} \end{pmatrix} \quad p+m+s=n$$

$$m(x,y) = x_1y_1 + \dots + x_py_p - x_{p+1}y_{p+1} - \dots - x_{p+m}y_{p+m}$$

p, m, s are invariants of M, do not dep. on the basis.

$\mathrm{rank}(M)=n \iff s=0$, can still have $+1, -1$ on diagonal

<u>Def</u> $m(x,y)$ is called <u>positive definite</u> if

$$m(x,x) > 0 \quad \text{for all } x \in V, \ x \neq 0$$

<u>Cor</u> $m(x,y)$ is <u>positive definite</u> iff we can choose a basis such that $M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, so $m(x,y) = x^Ty$ in this basis.

In this case, $\mathrm{Iso}(M) = O(n)$

<u>Ex</u> $M = \begin{pmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & \underbrace{+1}_{p} & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & \underbrace{-1}_{m} \end{pmatrix}$ $\quad \mathrm{Iso}(M) = O(p,m)$

$$\begin{array}{c} M = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & +1 & \\ & & & -1 \end{pmatrix} \\ \quad\quad\quad\; x \quad y \quad z \quad t \end{array}$$

$m(x,x) = x^2+y^2+z^2-t^2$ spacetime metric

$\mathrm{Iso}(M) = O(3,1)$ non-compact.

<u>M symmetric, $\mathbb{K}=\mathbb{C}$</u>

<u>Fact</u> We can change the basis to

$M = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \dots \end{pmatrix}$ $\quad$ M nondegenerate.

$$M = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

M nondegenerate

$$M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\mathrm{Iso}(M) = O(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : A^T A = I\}$$

Contains $O(n)$, <u>different</u> from $U(n)$!

<u>Rmk</u> $\boxed{O(n, \mathbb{C}) \cap U(n) = O(n)}$

$$\left.\begin{array}{ll} A^T A = I & O(n, \mathbb{C}) \\ \bar{A}^T A = I & U(n) \end{array}\right\}$$ these combined imply $A = \bar{A}$

$\Rightarrow A \in GL(n, \mathbb{R})$

$\Rightarrow A \in O(n) = O(n, \mathbb{R})$

M <u>antisymmetric</u> $m(x, y) = -m(y, x)$ for all x, y.

<u>Fact</u> In this case, we choose a basis in V such that

$$M = \begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & & & \\ & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & & \\ & & \ddots & & & \\ & & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

$M^T = -M$

M non-degenerate $\Rightarrow$ $M = \begin{array}{c} \begin{array}{cccccc} e_1 & f_1 & e_2 & f_2 & e_3 & f_3 \end{array} \\ \left(\begin{array}{cc|cc|cc} 0 & -1 & & & & \\ 1 & 0 & & & & \\ \hline & & 0 & -1 & & \\ & & 1 & 0 & & \\ \hline & & & & 0 & -1 \\ & & & & 1 & 0 \end{array}\right) \end{array}$ in particular $\dim V$ is even!

$m(e_i, f_j) = \delta_{ij}$

$m(f_j, e_i) = -\delta_{ij}$ $\quad e_1 \dots e_n, f_1 \dots f_n$

Can reorder the basis to get

$$J = \begin{array}{c} \begin{array}{cccccc} e_1 & \dots & e_n & f_1 & \dots & f_n \end{array} \\ \left(\begin{array}{ccc|ccc} & & & -1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -1 \\ \hline 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \end{array}\right) \end{array}$$

Def: The symplectic group $Sp(n, \mathbb{K}) = \{ A \in GL(2n, \mathbb{K}) : A^T J A = J \}$

Last time: $Sp(n, \mathbb{R}) = \{ A \in GL(2n, \mathbb{R}) : A^T J A = J \}$

Claim $Sp(n, \mathbb{R}) \cap O(2n) \simeq U(n)$

Cor $U(n) \subset Sp(n, \mathbb{R})$ as a subgroup.

Proof: $Sp(n, \mathbb{R})$: $A^T J A = J$

$O(2n)$: $\boxed{A^T A = I}$, $A^T = A^{-1}$

Combine these $\Rightarrow$ $A^{-1} J A = J$

$$\boxed{JA = AJ}$$

As we discussed last time, this implies

$$A = \left(\begin{array}{c|c} X & -Y \\ \hline Y & X \end{array}\right)$$

$$A = \left(\begin{array}{c|c} \cdot & \cdot \\ \hline Y & X \end{array}\right)$$

This is orthogonal iff the $n \times n$ matrix $\underbrace{X+iY}$ is unitary.

$$(\overline{X+iY})^T(X+iY) = (X^T - iY^T)(X+iY)$$

$$= \underbrace{(X^TX + Y^TY)}_{I} + i\underbrace{(X^TY - Y^TX)}_{0}$$

<u>Def</u> Compact symplectic group

$$Sp(n,\mathbb{C}) \cap U(2n) = Sp(n)$$

General construction

Given V, we can define the space of <u>all</u> bilinear forms $\mathcal{B}_V = (V \otimes V)^*$

This is a vector space of dimension n^2.

The group $GL(V)$ <u>acts</u> on the space $\mathcal{B}_V$

$$A \in GL(V),\ m \in \mathcal{B}_V \rightsquigarrow \tilde{m}(x,y) = m(A^{-1}x, A^{-1}y)$$

$$Iso(m) = \left\{\begin{matrix}\text{matrices}\\ \text{preserving}\end{matrix}\ m\right\} = Stab(m)$$

for this action!

<u>HW#4</u> Suppose that we have an continuous action of $GL(n)$ on some vector space $\mathcal{B}$

$Stab(v)$ is a matrix Lie group.