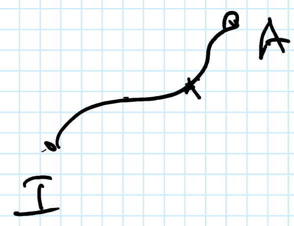
HW#3 G = matrix Lie group

G_0 = (path) connected component of I

$A \in G_0 \Rightarrow$ there is a continuous path $A(t)$ such that
- $A(t) \in G$ for all t
- $A(0) = I$
- $A(1) = A$



$B \in G_0 \Rightarrow$ path $B(t) \in G$ for all t

$B(0) = I$, $B(1) = B$

$A(t)B(t)$ = another path in G.
$A(0)B(0) = I \cdot I = I$
$A(1)B(1) = AB$

$\Big\}$ AB is connected to I by a path, $AB \in G_0$

Matrix exponential

Def X = any $n \times n$ matrix (real or complex)

$$e^X = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

Thm This is well defined, series converges for all X.

Proof: We define a norm on matrices (Hilbert-Schmidt)

$$\|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\mathrm{Tr}(A^*A)}$$

One can check that $\|AB\| \le \|A\| \cdot \|B\|$

$\|A+B\| \le \|A\| + \|B\|$

In particular, $\|X^n\| \le \|X\|^n$

Since $\displaystyle\sum_{n=0}^{\infty} \frac{\|X\|^n}{n!}$ converges for all $\|X\|$,

the series $\displaystyle\sum_{n=0}^{\infty} \frac{X^n}{n!}$ converges absolutely wrt this norm.

Note Converges uniformly on compacts $\|X\| \le K$

So e^X is a continuous (and smooth) function of X.

Properties 1) $e^0 = I$

2) If X and Y commute then

$$e^{X+Y} = e^X e^Y$$

(false if X, Y do not commute)

3) e^X is invertible, $(e^X)^{-1} = e^{-X}$.

Proof: ① Clear

② Assume $XY = YX$, then we claim that $(X+Y)^n = \displaystyle\sum_{a=0}^{n} \binom{n}{a} X^a Y^{n-a}$

$(X+Y) \cdot \ldots \cdot (X+Y)$ and expand

$$e^{X+Y} = \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = \sum_{n=0}^{\infty} \sum_{a=0}^{n} \frac{1}{\cancel{n!}} \cdot \frac{\cancel{n!}}{a!(n-a)!} X^a Y^{n-a} =$$

$$e^{X+Y} = \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = \sum_{n=0}^{\infty} \sum_{a=0}^{n} \frac{1}{n!} \frac{n!}{a!(n-a)!} X^a Y^{n-a} =$$

$$= \left(\sum_{a=0}^{\infty} \frac{X^a}{a!}\right) \cdot \left(\sum_{b=0}^{\infty} \frac{Y^b}{b!}\right).$$

(3) $e^X \cdot e^{-X} = e^{X-X} = e^0 = I$ since $X, -X$ commute.

How to compute e^X in practice?

(1) Change a basis to transform X into Jordan normal form:

$$CXC^{-1} = \begin{pmatrix} \lambda_1 & 1 & & & & & \\ & \ddots & 1 & & & 0 & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & 1 & & \\ & 0 & & & \ddots & 1 & \\ & & & & & \lambda_2 & \\ & & & & & & \ddots \end{pmatrix}$$

(2) If X is diagonal

$$X = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad e^X = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

$$X^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

If X diagonalizable: $CXC^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$X = C^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} C$$

$$e^X = C^{-1} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} C$$

$$e^X = C^{-1} e^{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}} C = C^{-1} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} C$$

(3) How to compute exp of Jordan block?

$$X = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \lambda \cdot I + N, \text{ where } N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

Since $\lambda \cdot I$ and N commute, we can write

$$e^X = e^{\lambda \cdot I} \cdot e^N = \underline{e^\lambda \cdot e^N}$$ need to compute this

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix} \qquad N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 0 \\ 0 & & & & & 0 \end{pmatrix}$$ and so on

and $N^n = 0$

$$e^N = I + N + \frac{N^2}{2!} + \dots + \frac{N^{n-1}}{(n-1)!} = \begin{pmatrix} 1 & 1 & 1/2! & 1/3! & \cdots & \\ & \ddots & \ddots & \ddots & \ddots & 1/3! \\ & & \ddots & \ddots & \ddots & 1/2! \\ & & & \ddots & \ddots & 1 \\ 0 & & & & & 1 \end{pmatrix}$$

Conclusion

$$\exp \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \begin{pmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2!} & \cdots & \frac{e^\lambda}{(n-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & e^\lambda/2! \\ & & & \ddots & e^\lambda \\ & & & & e^\lambda \end{pmatrix}$$

If we have several Jordan blocks we

If we have several Jordan blocks, we do it for each of them

And conjugate back to the original basis.

Fact $\boxed{\det(e^X) = e^{tr(X)}}$

Proof Change the basis,

$$X = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad e^X = \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

$$\det e^X = e^{\lambda_1} \cdot \ldots \cdot e^{\lambda_n} = e^{\lambda_1 + \ldots + \lambda_n} = e^{tr(X)}$$

Cor If X is a real matrix then $\det(e^X) > 0$!

Ex: $X = (it)$ $\quad e^X = (e^{it}) = (\cos t + i \sin t)$

$t \in \mathbb{R}$

$$X = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = t \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Exercise (HW) $e^X = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

Consider the family of matrices

$A(t) = e^{t \cdot X}$ $\quad X$ = fixed matrix $\quad t \in \mathbb{R}$

$$A(t) = e^{t\cdot X}, \quad X = \text{fixed matrix} \quad t \in \mathbb{R}$$

Facts ① $A(t_1 + t_2) = A(t_1)\cdot A(t_2)$

$A(-t) = A(t)^{-1}$

$A(0) = I$

Proof $t_1 X$ and $t_2 X$ commute $\Rightarrow$ $e^{(t_1 X + t_2 X)} = e^{t_1 X} e^{t_2 X}$

$$e^{(t_1 X + t_2 X)} = e^{(t_1+t_2)X}$$

② $\{A(t) : t \in \mathbb{R}\}$ is a subgroup of $GL(n)$!

It's called a one-parameter subgroup generated by X.

! **Warning** ! This subgroup is NOT necessary closed, so not necessary a matrix Lie group.

③ $\frac{d}{dt} A(t) = X A(t) = A(t)\cdot X$

$$A(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}$$

We can differentiate this term by term:

$$\frac{d}{dt} A(t) = \sum_{n=0}^{\infty} \frac{n t^{n-1} X^n}{n!} = X \sum_{n=0}^{\infty} \frac{t^{n-1} X^{n-1}}{(n-1)!} = X e^{tX} = X A(t).$$