$\pi_1(U(n)) = \mathbb{Z}$ $\quad \underbrace{\pi_1(SU(n)) = \{e\}}_{}$

SU(n) is simply connected!

Cor $GL(n,\mathbb{C})$ retracts onto $U(n) \Rightarrow \pi_1(GL(n,\mathbb{C})) = \mathbb{Z}$

$SL(n,\mathbb{C})$ retracts onto $SU(n) \Rightarrow \pi_1(SL(n,\mathbb{C})) = \{e\}$

One-parameter subgroups $\quad X =$ fixed matrix

$A(t) = e^{tX}$ – one-parameter subgroup

Recall:
- $A(t_1 + t_2) = A(t_1)A(t_2)$
- $A(-t) = A(t)^{-1}$
- $\frac{d}{dt} A(t) = XA(t)$

ODE perspective $\quad v(t) \in \mathbb{C}^n$

Want to solve linear ODE:

$$\frac{d}{dt} v(t) = X v(t), \quad v(0) = v_0$$

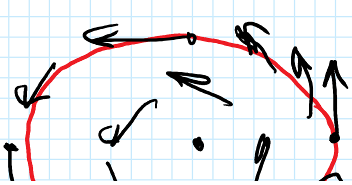
Solution: $v(t) = e^{tX} \cdot v_0 = A(t) v_0$

Vector field Xv at every point $v \in \mathbb{C}^n$

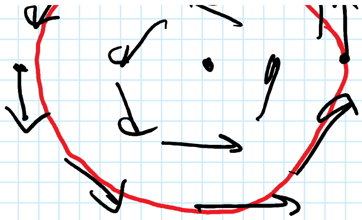
$v(t)$ is tangent to this vector field at every point

Ex $\mathbb{R}^2$ $\quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

draw a vector $(-x_2, x_1)$ at every point (x_1, x_2)



$v(t) = e^{tX} \cdot v_0$



$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} v_0$$

HW3

Can look at matrix ODE:

$$\frac{d}{dt} A(t) = XA(t) \quad , \quad A(0) = I \qquad (*)$$

equivalent to n ODE w. matrix X
for column of $A(t)$.

Note: by uniqueness of solutions of ODE,
$A(t)$ is uniquely determined by $(*)$.

Rmk We can say that there's a vector field on $Mat(n \times n)$
vector XA at a point $A \in Mat(n \times n)$.

Rmk In fact, we can consider this as a vector field on $GL(n)$.

Ex $X = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \alpha \end{pmatrix}$ $\alpha \in \mathbb{R}$ fixed real number

$$A(t) = \exp(tX) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{2\pi i \alpha t} \end{pmatrix}$$

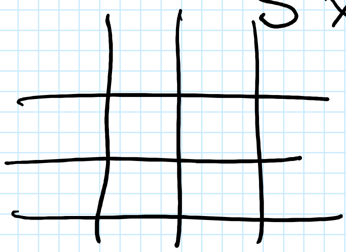
What can we say about this?

- Always contained in

$$U(1)\times U(1) = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{2\pi i b} \end{pmatrix} \quad a,b\in\mathbb{R}$$

$$\Vert$$

$$S^1\times S^1 = T^2 = \underset{(a,b)}{\mathbb{R}^2}/\mathbb{Z}^2$$



$$(a,b) \longrightarrow (a+k,\ b+\ell) \quad k,\ell\in\mathbb{Z}$$

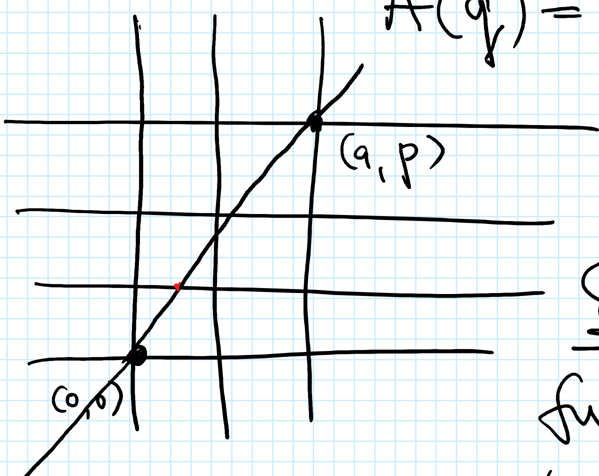
$$e^{2\pi i(a+k)} = e^{2\pi i a}\cdot e^{2\pi i k} = e^{2\pi i a}$$

<u>Case 1</u> $\alpha\in\mathbb{Q}$

$$\alpha = \frac{p}{q}$$

$$A(t) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{2\pi i \alpha t} \end{pmatrix}$$

$$A(q) = \begin{pmatrix} e^{2\pi i q} & 0 \\ 0 & e^{2\pi i \frac{p}{q}\cdot q} \end{pmatrix} = \begin{pmatrix} e^{2\pi i q} & 0 \\ 0 & e^{2\pi i p} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



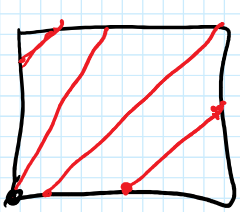
<u>So:</u> $A(t)$ is a periodic function of t, $A(t+q) = A(t)$

<u>Note:</u> $A(t): (\mathbb{R},+) \longrightarrow U(1)\times U(1)$

group homomorphism

<u>not</u> injective

image is closed



$\Rightarrow$ image = matrix Lie group.

<u>Case</u> $\alpha\notin\mathbb{Q} \Rightarrow$ all matrices $A(t)$ are distinct

for different t,

and we have injective homomorphism $(\mathbb{R},+) \to U(1) \times U(1)$

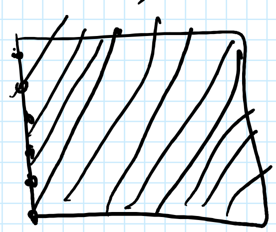
<u>Proof</u>: $A(t_1) = A(t_2)$

$$\begin{pmatrix} e^{2\pi i t_1} & 0 \\ 0 & e^{2\pi i \alpha t_1} \end{pmatrix} = \begin{pmatrix} e^{2\pi i t_2} & 0 \\ 0 & e^{2\pi i \alpha t_2} \end{pmatrix}$$

$t_1 - t_2 \in \mathbb{Z}$ $\quad \alpha \notin \mathbb{Q} \Rightarrow \underline{t_1 - t_2 = 0}$.

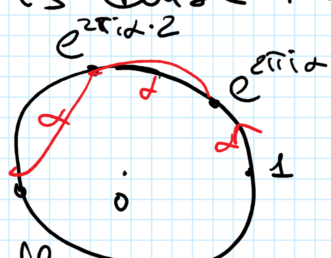
$\alpha(t_1 - t_2) \in \mathbb{Z}$

<u>Fact</u> (HW3) $A(t)$ is <u>dense</u> in $T^2 = U(1) \times U(1)$



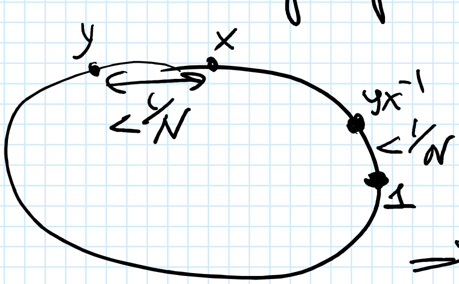
<u>Sketch of the proof</u>

① $\{e^{2\pi i \alpha k} : k \in \mathbb{Z}\} = \Gamma_\alpha$ is dense in S^1 (if $\alpha \notin \mathbb{Q}$)



<u>Hint</u>: there are 2 points in Γ_α with difference of angles $< \frac{1}{N}$

Γ_α is a subgroup of S^1!



$\Rightarrow$ there is a point yx^{-1} with argument $< \frac{1}{N}$

$\Rightarrow$ use this to find to approximate every point in S^1.

every point in S^1.

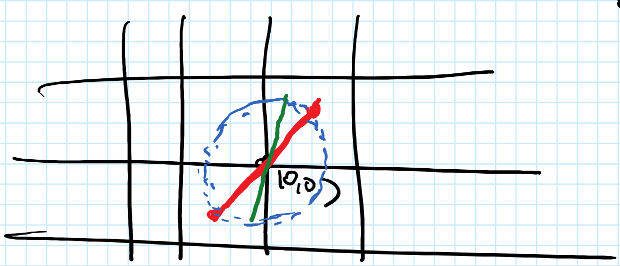
(2) $t = k \in \mathbb{Z}$ $\quad \begin{pmatrix} e^{2\pi i k} & 0 \\ 0 & e^{2\pi i \alpha k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \alpha k} \end{pmatrix}$

dense in S^1 by (1)

In particular, if $\alpha \notin \mathbb{Q} \Rightarrow A(t)$ is not closed!

This is an example of continuous homomorphism between Lie groups where image is not closed, and hence not a Lie group.

But locally $\{A(t) : |t| < \varepsilon\}$ is nice!



- this is a piece of straight line in some neighborhood of $e^{0 \cdot X} = I$
- It is closed in this neighborhood

- By varying α (or X in general) we can cover the whole neighborhood of I
- Homeomorphism between the neighborhood of $I \in U(1) \times U(1)$ and neighborhood of $(0,0) \in \mathbb{R}^2$

$(0,0) = (a,b)$

of $I \in U(1)\times U(1)$ and neighbourhood of $(0,0)$.

$$U(1)\times U(1) \xleftarrow{e^X} (0,0) = (a,b)$$

- <u>Note</u> $\{A(t) : |t| < \varepsilon\}$ generates the whole one-parameter subgroup.