Lecture 8 (4/15)

Friday, April 15, 2022 1:07 PM

Matrix logarithm

Lemma $\|e^X - I\| \le e^{\|X\|} - 1$

Proof: $e^X - I = X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$

$\|e^X - I\| \le \|X\| + \frac{\|X\|^2}{2!} + \frac{\|X\|^3}{3!} + \dots = e^{\|X\|} - 1$ ∎

A = matrix, we define

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m} \quad (*)$$

Recall (**) $\ln x = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(x-1)^m}{m}$ ← power series w. center at $x=1$

$\ln(1+z) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{z^m}{m}$ ← power series w. center at $z=0$

Fact from calculus: radius of convergence is 1

So series (**) for $\ln(x)$ actually converges and defines $\ln(x)$ for $|x-1| < 1$

But when it converges, $e^{\ln x} = \ln(e^x) = x$.

Thm 2.8 Define $\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$

① This series converges for $\|A-I\| < 1$ and for such A the sum is a continuous function of A

② If $\|A-I\|<1$ then $e^{\log A}=A$

③ If $\|X\|<\log 2$ then

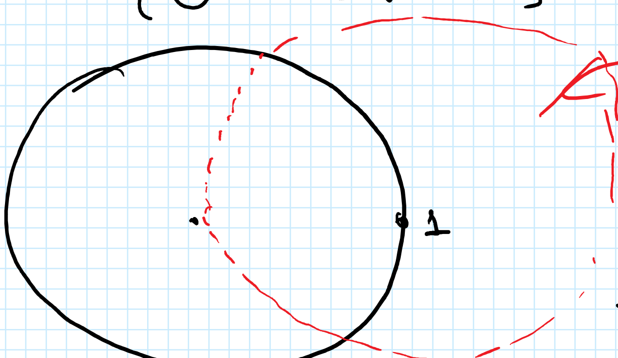
$$\|e^X - I\| \underset{\text{by Lemma}}{\leq} e^{\|X\|}-1 \leq e^{\log 2}-1 = 2-1 = 1.$$

and in this case $\log(e^X)=X$.

<u>Proof</u>: Skip, see Th 2.8 in the book.

<u>Rmk</u> For some A with $\|A-I\|>1$ the series for $\log A$ might converge (ex. if $A-I$ nilpotent) but it is not clear if ② and ③ are satisfied.

<u>Ex</u> $U(1)=\{e^{2\pi i t} : t\in\mathbb{R}\}$



$\log A$ is defined in circle w. center at 1 and radius 1

If $|e^{2\pi i t}-1|<1$ then $\log(e^{2\pi i t})=2\pi i t$

<u>Lie product formula</u>

$X, Y =$ matrices

$m\in\mathbb{Z}_{>0}$

$$e^{X+Y}=\lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m$$

<u>Remark</u>

$e^{X+Y}\neq e^X e^Y$

Proof $e^{\frac{X}{m}} = I + \frac{X}{m} + O(($

<u>Proof</u> $e^{\frac{X}{m}} = I + \frac{X}{m} + O\left(\frac{1}{m^2}\right)$

$e^{X+Y} \neq e^X e^Y$ if X and Y do not commute

$e^{\frac{Y}{m}} = I + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = \left(I + \frac{X}{m} + O\left(\frac{1}{m^2}\right)\right)\left(I + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right)$$

$$= I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$$

$$\log\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right) = \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} - I\right) - \frac{\left(e^{\frac{X}{m}} e^{\frac{Y}{m}} - 1\right)^2}{2} + \dots =$$

$$= \left(\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right) - \frac{\left(\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right)^2}{2} + \dots =$$

$$= \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$$

$$\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = \exp\left(m \cdot \log\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)\right) =$$

$$= \exp\left(m \cdot \left(\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right)\right) = \exp\left(X + Y + O\left(\frac{1}{m}\right)\right)$$

As $m \to \infty$, $O\left(\frac{1}{m}\right) \to 0$, and exp is continuous

so $\lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = e^{X+Y}$. $\blacksquare$

Suppose that G = matrix Lie group.

Def The **Lie algebra** of G is the set of all matrices X such that $\boxed{e^{tX} \in G \text{ for all } t \in \mathbb{R}}$!

$$\mathfrak{g} = \mathrm{Lie}(G)$$

Ex $U(1) = \{ e^{it} : t \in \mathbb{R} \} \subset GL(1; \mathbb{C})$

$$\mathrm{Lie}\, U(1) = \{ z \in \mathbb{C} : e^{tz} \in U(1) \text{ for all } t \}$$

$z = x + iy$

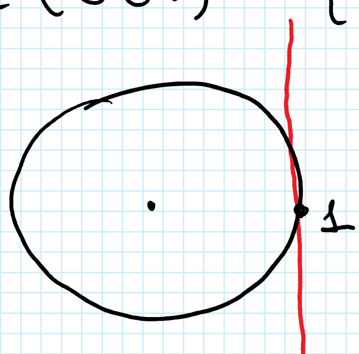
$$e^{tz} = e^{tx + ity} = e^{tx} \cdot (\cos(ty) + i \sin(ty))$$

So we want $e^{tx} = 1 \iff tx = 0$ for all t

$\iff x = 0$

$\Rightarrow z$ is purely imaginary.

$\mathrm{Lie}(U(1)) = \{ iy : y \in \mathbb{R} \}$ = the set of imaginary numbers.



Note The tangent space to $U(1) = S^1$ at $1 \simeq \mathrm{Lie}(U(1))$ = set of all imaginary numbers.

By the above, exp and log give an explicit identification of the neighborhood of $1 \in U(1)$ and a neighborhood of $0 \in \mathrm{Lie}(U(1))$

$G = O(n) = \{ A \in GL(n, \mathbb{R}) : A^T A = I \}$

What is $\mathrm{Lie}(G)$? We need to find all X such that $e^{tX} \in O(n)$.

<u>Answer:</u> $\mathrm{Lie}(G) = \{$ skew-symmetric matrices $X^T = -X \}$

① Suppose $X^T = -X$

$$e^{X^T} = (e^X)^T = e^{-X} = (e^X)^{-1}$$

So $(e^X)^T = (e^X)^{-1}$ and $(e^X)^T \cdot e^X = I$.

If $X^T = -X$ then $tX^T = -tX$ and the same applies to tX.

② Suppose $e^{tX} = A(t)$ is orthogonal for all t

$$A(t)^T A(t) = I$$

Take derivative in t!

$$A'(t)^T \cdot A(t) + A(t)^T \cdot A'(t) = 0$$

Plug in $t=0$ into derivatives:

$$A(0) = I \qquad A'(0) = X$$

So we get

$X^T \cdot I + I^T \cdot X = 0$ and $X^t + X = 0$.