G = matrix Lie group

Def Lie algebra of $G = \mathrm{Lie}(G) = \mathfrak{g}$ is the set of all matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

Thm (3.20) in book. Lie algebra of G satisfies the following:

① If $X \in \mathrm{Lie}(G)$, then $sX \in \mathrm{Lie}(G)$ for all $s \in \mathbb{R}$

② If $X, Y \in \mathrm{Lie}(G)$, then $X+Y \in \mathrm{Lie}(G)$

$\Rightarrow$ Lie(G) is a real vector subspace of Mat($n \times n$)

③ If $A \in G$ and $X \in \mathrm{Lie}(G)$ then $AXA^{-1} \in \mathrm{Lie}(G)$

G acts on Lie(G) adjoint representation

④ If $X, Y \in \mathrm{Lie}(G)$ then $XY - YX \in \mathrm{Lie}(G)$

Proof ① $e^{t\cdot(sX)} = e^{ts\cdot X} \in G$ ✓

② Need to check $e^{t(X+Y)} \in G$ for all t

$$e^{t(X+Y)} = e^{tX+tY} \underset{\substack{\text{Lie product}\\\text{formula from last lecture}}}{=\!=\!=} \lim_{m\to\infty}\left(e^{\frac{tX}{m}} \cdot e^{\frac{tY}{m}}\right)^m$$

Now $e^{\frac{tX}{m}} \in G$, $e^{\frac{tY}{m}} \in G \Rightarrow e^{\frac{tX}{m}} \cdot e^{\frac{tY}{m}} \in G$

$\Rightarrow \left(e^{\frac{tX}{m}} \cdot e^{\frac{tY}{m}}\right)^m \in G$

Since G is <u>closed</u> in $GL(n)$, $\lim_{m\to\infty}(\cdots)$ is invertible

$\Rightarrow \lim_{m\to\infty}(\cdots) \in G$.

(3) We want to prove $e^{t\cdot AXA^{-1}} \in G$

$$e^{t\cdot AXA^{-1}} = \boxed{e^{A\cdot tX\cdot A^{-1}} = Ae^{tX}A^{-1}}$$

Now $e^{tX} \in G, A \in G \Rightarrow Ae^{tX}A^{-1} \in G$

(4) Want to prove $XY - YX \in Lie(G)$

Define $A(t) = e^{tX} \in G$

By (3) $A(t)\cdot Y\cdot A(t)^{-1} \in Lie(G)$ for all t

Take derivative in t and use product rule:

$$\frac{d}{dt}\left[A(t)YA(t)^{-1}\right] = A'(t)YA(t)^{-1} + A(t)\cdot Y\cdot\left(A(t)^{-1}\right)' \quad (*)$$

$A(t)^{-1} = e^{-tX} \Rightarrow$ know the derivative of $A(t)^{-1}$

Plug in $t=0$: $A(0) = I$, $A^{-1}(0) = I$

$A'(0) = X$, $\left(A^{-1}(t)\right)'\Big|_{t=0} = -X$

$(*)$ at $t=0$: $\frac{d}{dt}\left[A(t)YA(t)^{-1}\right]_{t=0} = XY\cdot I + I\cdot Y(-X)$
$= XY - YX$.

Conclusion: $A(t)YA^{-1}(t) \in Lie(G)$ = vector subspace of $Mat(n\times n)$ by (1) and (2)

$$\frac{d}{dt}(\cdots)\Big|_{t=0} = \lim_{t\to 0}\frac{A(t)YA^{-1}(t) - Y}{t} \in Lie(G).$$

Def An (abstract) Lie algebra is a vector space $\mathfrak{g}$ with a map $[\,,\,]: \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$

$(X,Y) \longrightarrow [X,Y]$ commutator

satisfying the following properties:

- $[\ ,\ ]$ is bilinear in X and Y
- Skew-symmetric $[X,Y] = -[Y,X]$

Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$$\Leftrightarrow \quad [X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

Key example A = associative algebra (ex. $\mathrm{Mat}(n\times n)$), define

$$[X,Y] = XY - YX.$$

Claim: this gives A the structure of a Lie algebra

- bilinear, skew-symmetric $\longrightarrow$ clear
- $[X,YZ] = [X,Y]Z + Y[X,Z] \quad (**)$

Proof: $LHS = XYZ - YZX$

$RHS = XYZ - YXZ + YXZ - YZX$ ✓

Now

$$[X,[Y,Z]] = [X, YZ - ZY] = (**)$$

$$[X,Y]Z + Y[X,Z] - [X,Z]Y - Z[X,Y]$$

$$= [[X,Y],Z] + [Y,[X,Z]].$$

∎

Cor Suppose $\mathfrak{g} \subset \mathrm{Mat}(n\times n)$ such that

- $\mathfrak{g}$ is a vector subspace
- For X,Y in $\mathfrak{g}$ we have $[X,Y] \in \mathfrak{g}$.

Then $\mathfrak{g}$ is a Lie algebra.

Cor G = matrix Lie group then by Thm 3.20 $\mathrm{Lie}(G)$ is a Lie algebra.

What info do we need to define a Lie algebra:

- Choose a basis $x_1 \dots x_d$ in $\mathfrak{g}$

$$[x_i, x_j] = \sum_k c_{ij}^k x_k \qquad c_{ij}^k = -c_{ji}^k \quad + \text{Jacobi identity.}$$

__Examples__ a) $G = GL_n$ $\quad \boxed{\mathrm{Lie}(G) = \mathrm{Mat}(n \times n)} = \mathfrak{gl}_n$

(e^{tX} is invertible).

b) $G = SL_n$ (det=1) $\quad \boxed{\mathrm{Lie}(G) = \{X : \mathrm{Tr}(X) = 0\}} = \mathfrak{sl}_n$

$$\det e^{tX} = e^{\mathrm{Tr}(tX)} = e^{t \cdot \mathrm{Tr}(X)}$$

$$\det e^{tX} = 1 \Rightarrow t \cdot \mathrm{Tr}(X) = 2\pi i k \quad k \in \mathbb{Z}.$$

$$\Rightarrow \underline{\underline{\mathrm{Tr}(X) = 0}}.$$

__Ex__ Lie algebra $\mathfrak{sl}_2 = \{2\times 2 \text{ matrices}, \mathrm{Tr} = 0\}$

__Basis:__ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

3 dimensional vector subspace of $\mathrm{Mat}(2\times 2)$.

$[H, E]$, $[H, F]$, $[E, F]$ $\longleftarrow$ __homework__

c) $G = GL_n^+ = \{A \in GL_n(\mathbb{R}) : \det A > 0\}$

$$\mathrm{Lie}(GL_n^+) = \mathfrak{gl}_n = \mathrm{Lie}(GL_n)$$

$$\det e^{tX} = e^{t \cdot \mathrm{Tr} X} > 0$$