MAT 261A, Spring 2022
Homework 5

Due before 1:10 on Wednesday, May 4

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

Recall that the Lie algebra $\mathfrak{sl}_2$ has the basis E, F, H and relations

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H.$$

1. Let $\lambda \in \mathbb{C}$. The *Verma module* $\Delta(\lambda)$ is an infinite-dimensional representation of $\mathfrak{sl}_2$ with basis $v_0 = v, v_1 = Fv, v_2 = F^2v, v_3 = F^3v, \ldots$ such that

$$Ev = 0 \text{ and } Hv = \lambda v$$

$$\begin{tikzcd} 0 & v_0 \arrow[l, "E"'] \arrow[r, bend left, "F"] & v_1 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & v_2 \arrow[l, bend left, "E"] \arrow[r, bend left, "F"] & \ldots \arrow[l, bend left, "E"] \end{tikzcd}$$

Clearly, $Fv_k = v_{k+1}$.

(a) Find Ev_k and Hv_k for all k.

(b) Suppose that λ is not a nonnegative integer, prove that $\Delta(\lambda)$ is irreducible.

(c) Suppose that $\lambda = m$ is a nonnegative integer. Prove that $\Delta(m)$ has a unique proper submodule R_m, and the quotient $L(m) = \Delta(m)/R_m$ is finite-dimensional. Find the dimension of $L(m)$.

2. Prove that $L(0)$ is trivial and $L(1)$ is isomorphic to the vector representation $\mathbb{C}^2$.

3. Prove that $L(2)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2$.

4. Suppose that $V = \oplus_m L(m)^{\oplus k_m}$, let a_i denote the dimension of the H-eigenspace in V with eigenvalue i. Prove that $a_i = a_{-i}$, $a_i \geq a_{i+2}$ for $i \geq 0$ and $a_i \geq a_{i-2}$ for $i \leq 0$.