Generic curves and non-coprime Catalans

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Compactified Jacobians

Let C be a **reduced** and **irreducible** complex plane curve singularity. We can parametrize C by $(x(t), y(t))$, then the completed local ring $\mathcal{O}_{C,0}$ has the form

$$\mathcal{O}_{C,0} = \mathbb{C}[[x(t), y(t)]]$$

Definition

The (local) compactified Jacobian $\overline{JC}$ is the moduli space of $\mathcal{O}_{C,0}$-submodules of $\mathbb{C}[[t]]$.

Equivalently, this is is the space of rank 1 torsion free-sheaves on C with an appropriate choice of framing.

Example

For example, consider the curve $C = \{x^2 = y^3\}$ with parametrization $x(t) = t^3, y(t) = t^2$. Then $\overline{JC}$ parametrizes subspaces $M \subset \mathbb{C}[[t]]$ such that

$$t^2 M \subset M, t^3 M \subset M.$$

Up to multiplication by a power of t, there are two types of subspaces:

$$\langle 1 + \lambda t, t^2, t^3, t^4, \ldots \rangle, \langle 1, t, t^2, t^3, t^4, \ldots \rangle$$

Here $\lambda \in \mathbb{C}$ is an arbitrary parameter, and $\overline{JC} = \mathbb{C} \sqcup \{pt\} = \mathbb{CP}^1$.

Compactified Jacobians

- (Altman, Kleiman, Iarrobino) $\overline{JC}$ is irreducible, $\dim \overline{JC} = \delta(C)$ (delta invariant)
- (Altman, Kleiman, Iarrobino) For $N \gg 0$, the Hilbert schemes $\mathrm{Hilb}^N(C,0)$ stabilize to $\overline{JC}$
- $\overline{JC}$ is related to the singular fiber in the **Hitchin fibration**
- $\overline{JC}$ is isomorphic to the **affine Springer fiber** for an element $\gamma \in GL_n((t))$ such that the characteristic polynomial of γ agrees with the equation of C
- The number of points in $\overline{JC}$ over a finite field $\mathbb{F}_q$ is closely related to **orbital integrals** in number theory.
- Cherednik and Danilenko conjectured a formula for the Poincaré polynomials of $\overline{JC}$ in terms of Macdonald polynomials and double affine Hecke algebras (DAHA).

Compactified Jacobians

One recent breakthrough is the following result.

Theorem (Kivinen-Tsai '22)

The number of points in $\overline{JC}$ over a finite field $\mathbb{F}_q$ is a polynomial in q with nonnegative coefficients.

The proof uses orbital integrals and p-adic harmonic analysis.

Kivinen and Tsai also proved a combinatorial formula for these polynomials in terms of Puiseaux expansion of C, and proved that it agrees with the Cherednik-Danilenko conjecture. In particular, it follows that the latter is indeed a polynomial with nonnegative coefficients.

Problem (Goresky-Kottwitz-MacPherson?)

For which C the compactified Jacobian $\overline{JC}$ admits an affine paving?

Compactified Jacobians

Let K be the knot obtained by the intersection of C with a small sphere centered at the origin. For example, for $C = \{x^2 = y^3\}$ we get the trefoil knot.

Conjecture (Oblomkov, Rasmussen, Shende)

The **Khovanov-Rozansky homology** *of K is isomorphic to*

$$\bigoplus_{n=0}^{\infty} H^*(\mathrm{Hilb}^n(C,0)).$$

In particular, the homology of the Hilbert schemes of points on C is expected to be determined by the topological type of K (or, equivalently, by the Puiseaux expansion of C), which is not obvious at all from the algebro-geometric point of view.

By considering $\mathrm{Hilb}^N(C,0)$ for large N, we arrive at:

Conjecture (Weak ORS conjecture)

The homology of $\overline{JC}$ is determined by the reduced Khovanov-Rozansky homology of K (and, in particular, by the topological type of K).

Main results

Now we can state our main results.

Definition

Consider a curve C defined by the parametrization

$$\begin{cases} x(t) = t^{nd} \\ y(t) = t^{md} + \lambda t^{md+1} + \dots \end{cases}$$

where $GCD(m,n) = 1$. We call C **generic** if either $d = 1$, or $\lambda \neq 0$.

Such C corresponds to the $(d, mnd+1)$ cable of the (m,n) torus knot. For $d = 1$ we simply get a (m,n) torus knot.

Main results

Theorem (G., Mazin, Oblomkov)

Suppose that C is generic. Then $\overline{JC}$ admits an affine paving, the cells are parametrized by the lattice paths in the $(md) \times (nd)$ rectangle below the diagonal, and we give a combinatorial formula for the dimension of such a cell.

For $d = 1$ the affine paving was constructed earlier by Piontkowski.

Corollary

For generic curves, the homology of $\overline{JC}$ is supported in even degrees and does not depend on λ or higher order terms in $y(t)$.

Main results

Also, we check that our expression for the Poincaré polynomial of $\overline{JC}$ agrees with the formulas of Kivinen-Tsai and Cherednik-Danilenko.

Corollary

The Euler characteristic $c_{md,nd} = \chi(\overline{JC})$ is given by the number of lattice paths in the $(md) \times (nd)$ rectangle below the diagonal. It is given by the generating function [Bizley, Grossman]

$$1 + \sum_{d=1}^{\infty} z^d c_{md,nd} = \exp\left[\sum_{d=1}^{\infty} \frac{(md+nd-1)!}{(md)!(nd)!} z^d\right].$$

For the curve $(t^4, t^6 + \lambda t^7 + \ldots)$ we get 23 cells labeled by Dyck paths in 4×6 rectangle.

Main results

The connection to knot homology is also true for generic curves.

Theorem (Hogancamp, Mellit)

Khovanov-Rozansky homology of the links of torus knots ($d = 1$) is supported in even degrees and the triply graded Poincaré polynomial is given by an explicit combinatorial formula.

Theorem (Caprau, Gonzalez, Hogancamp, Mazin, in progress)

The weak ORS conjecture holds for generic curves. In particular, Khovanov-Rozansky homology of the links of generic curves is supported in even degrees and the triply graded Poincaré polynomial is given by an explicit combinatorial formula.

Main results

Now we discuss the strategy of proof. Given a function $f(t) \in \mathbb{C}[[[t]]$, we write $\mathrm{Ord}(f) = n$ if

$$f(t) = at^n + \text{higher order terms}, \ a \neq 0$$

Given $M \subset \mathbb{C}[[t]]$, we define a subset

$$\Delta_M = \{\mathrm{Ord}(f) | f \in M\} \subset \mathbb{Z}_{\geq 0}.$$

Now we can consider strata in the compactified Jacobian with fixed Δ_M:

$$\Sigma_\Delta = \{M \in \overline{JC} : \Delta_M = \Delta\}.$$

Clearly, $x(t)M \subset M, y(t)M \subset M$ imply $\Delta + nd \subset \Delta$ and $\Delta + md \subset \Delta$. But there are more subtle conditions.

Main results

Example

Consider the curve $(t^4, t^6 + \lambda t^7)$ and $\Delta = \{0, 2, 4, 6, 8, 10, 11, 12, 13, \ldots\}$. Let us look for M such that $\Delta_M = \Delta$. We have

$$f_0 = 1 + at + \ldots \in M, \;\; f_2 = t^2 + bt^3 + \ldots \in M.$$

Now

$$y(t)f_0 - x(t)f_2 = (t^6 + \lambda t^7)(1 + at + \ldots) - t^4(t^2 + bt^3 + \ldots) = (a + \lambda - b)t^7 + \ldots$$

$$x(t)^2 f_0 - y(t)f_2 = t^8(1 + at + \ldots) - (t^6 + \lambda t^7)(t^2 + bt^3 + \ldots) = (a - \lambda - b)t^9 + \ldots$$

But by our choice of Δ there are no functions of order 7 or 9 in M, so

$$a + \lambda - b = a - \lambda - b = 0.$$

Since $\lambda \neq 0$, we get a contradiction.

Main results

Motivated by this example and its generalizations, we introduce the class of **admissible** subsets Δ and prove the following:

Theorem (G., Mazin, Oblomkov)

1. *If Δ is not admissible, then it does not correspond to a module M, and the corresponding stratum Σ_M is empty.*
2. *If Δ is admissible, then Σ_M is isomorphic to an affine space $\mathbb{C}^{\dim(\Delta)}$ where*
$$\dim(\Delta) = \sum_{i=0}^{nd} [a_i, a_i + md[\setminus \Delta$$
where a_i are the (nd)-generators of Δ.
3. *There is a bijection between the admissible subsets and the lattice paths in $(md) \times (nd)$ rectangle below the diagonal.*

In fact, $\dim(\Delta)$ is the number of free parameters minus the number of defining equations of Σ_Δ.

Example

Consider the curve $(t^6, t^9 + \lambda t^{10})$ and $\Delta = \{0, 3, 6, 7, 9, 10, 12, 13, 15, \ldots\}$. We have generators

$$f_0 = 1 + a_1 t + a_2 t^2 + \ldots, \;\; f_3 = t^3 + b_1 t^4 + b_2 t^5 + \ldots,$$

$$f_7 = t^7 + c_1 t^8 + \ldots, \;\; f_{10} = t^{10} + d_1 t^{11} + \ldots$$

Here a_i, b_i, c_i, d_i are free parameters. Two of the equations are

$$a_2 + \lambda a_1 - b_2 - (a_1 + \lambda - b_1) d_1 = b_2 + \lambda b_1 - a_2 - (b_1 + \lambda - a_1) c_1 = 0$$

If we add the equations and divide by λ, we get

$$a_1 + b_1 = \frac{1}{\lambda}((a_1 + \lambda - b_1) d_1 + (b_1 + \lambda - a_1) c_1)$$

which can be solved after a change of variables $(a_1 + b_1, a_1 - b_1, a_2 - b_2)$.

Beyond generic curves

We can also consider more general curves with two Puiseaux pairs:

$$\begin{cases} x(t) = t^{nd} \\ y(t) = t^{md} + \lambda t^{md+s} + \dots \end{cases}$$

where $GCD(m,n) = 1$ and $GCD(d,s) = 1$. We introduce the notion of s-**admissible** subsets.

Conjecture (G.,Mazin,Oblomkov)

1. *If Δ is not s-admissible then $\chi(\Sigma_\Delta) = 0$.*
2. *If Δ is s-admissible then $\chi(\Sigma_\Delta) = 1$.*

Note that there are examples when Σ_Δ is not an affine space.

Corollary

Assuming the conjecture, $\chi(\overline{JC})$ equals the number of s-admissible subsets.

Beyond generic curves

Definition

A cabled Dyck path with parameters $(n, m), (d, s)$ is the following collection of data:

- A (d, s) Dyck path P
- The numbers $v_1(P), \ldots, v_k(P)$ parametrizing the lengths of vertical runs in P, so that

$$v_1(P) + \ldots + v_k(P) = d$$

- An arbitrary tuple of $(v_i(P)n, v_i(P)m)$-Dyck paths.

For $s = 1$ there is only one (d, s) Dyck path with $v_1 = d$, and cabled Dyck paths are just (dm, dn) Dyck paths.

Beyond generic curves

Theorem (Kivinen, Tsai)

The Euler characteristic $\chi(\overline{JC})$ equals the number of cabled Dyck paths. In fact, the point count for $\overline{JC}(\mathbb{F}_q)$ is given by an explicit sum over cabled Dyck paths.

We proved that our conjecture is compatible with the above results.

Theorem (G.,Mazin,Oblomkov)

There is a bijection between s-admissible subsets and cabled Dyck paths.

Thank you!