Triply graded link homology: results and structures

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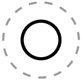
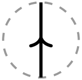
Khovanov and Rozansky defined in 2005 a triply graded link homology theory which generalizes HOMFLY-PT polynomial. In this talk, I will describe the progress in understanding this homology, focusing on:

- Examples of computations of Khovanov-Rozansky homology (joint with Alex Chandler, in progress)
- General structures in Khovanov-Rozansky homology (joint with Matt Hogancamp and Anton Mellit)
- Geometric models for some classes of links (joint with Roger Casals, Mikhail Gorsky and José Simental)

HOMFLY-PT invariant

The HOMFLY-PT invariant of links is defined by the following rules:

 $-$  $= (q - q^{-1})$  .

 $= \dfrac{a - a^{-1}}{q - q^{-1}}$  ,  $= -a^{-1}$ 

Given an (oriented) link diagram in the plane, we can use these rules to simplify it until the link becomes trivial.

HOMFLY homology

Khovanov and Rozansky defined HOMFLY homology and proved that it is a link invariant. To any link they assign a triply graded vector space $\mathcal{H} = \bigoplus_{i,j,k} \mathcal{H}_{i,j,k}$ such that the graded Euler characteristic

$$\sum_{i,j,k} q^i a^j (-1)^k \dim \mathcal{H}_{i,j,k} = P(a,q)$$

recovers the HOMFLY-PT polynomial.

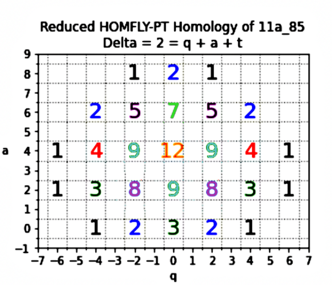
The definition of Khovanov-Rozansky homology is quite involved and uses **Hochschild homology** for complexes of **Soergel bimodules**. We will not need it.

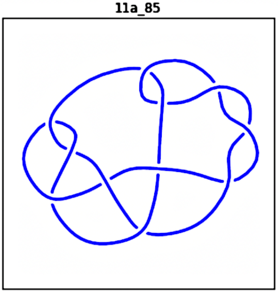
HOMFLY-PT homology: examples

- It is easy to compute the HOMFLY homology for 2-strand braids and $T(2, n)$ torus links by hand
- Direct computation of HOMFLY homology for any other example is hard! It involves a complex of modules over a polynomial ring with the number of terms exponential in the number of crossings.
- Nevertheless, Nakagane and Sano recently computed HOMFLY homology for all knots with at most 10 crossings, and most 11-crossing knots, so there is a lot of data to be explored. Alex Chandler and I are currently working on this.

HOMFLY-PT homology: examples

The easiest way to visualize data is using (q, a) grading from HOMFLY polynomial, and δ-grading. In some normalization, $\delta = a + q + t$. Here's an example of a two-bridge knot $11a_{85}$:



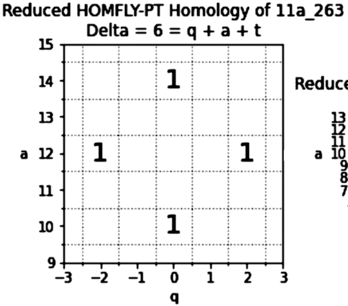


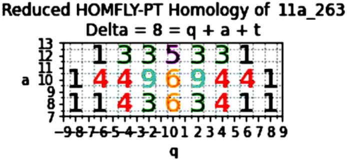
Theorem (Rasmussen,2007)

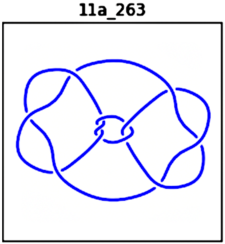
HOMFLY homology for all two-bridge knots is supported in a single δ-grading. As a consequence, it is determined by the HOMFLY polynomial.

HOMFLY-PT homology: examples

For most alternating knots in the dataset, HOMFLY homology is supported in a single δ-grading. The only exception is the knot $11a_{263}$:



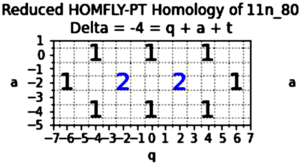


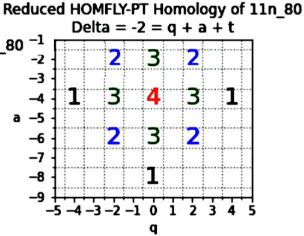


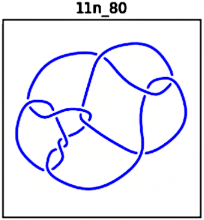
Open problem: what makes it special? How to compute HOMFLY homology of alternating links in general?

HOMFLY-PT homology: examples

All knots in the dataset are supported in at most two δ-gradings. Here is a bigger example (non alternating, not two-bridge) supported in two :







In the dataset we have the following statistics:

	$\Delta = 1$	$\Delta = 2$
two-bridge	173	0
alternating, not two-bridge	293	1
other	137	91
Total	603	92

HOMFLY-PT homology: examples

A different source of examples comes from torus knots. Here is the $(3, 4)$ torus knot[1] and its HOMFLY homology[2]:

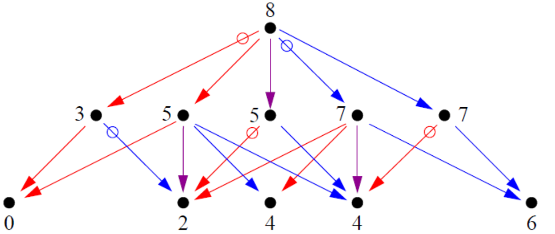


FIGURE 3.7. Differentials for $T_{3,4}$. The bottom row of dots has a-grading 6. The leftmost dot on that row has q-grading -6, which you can determine by noting that the vertical axis of symmetry corresponds to the line $q = 0$.

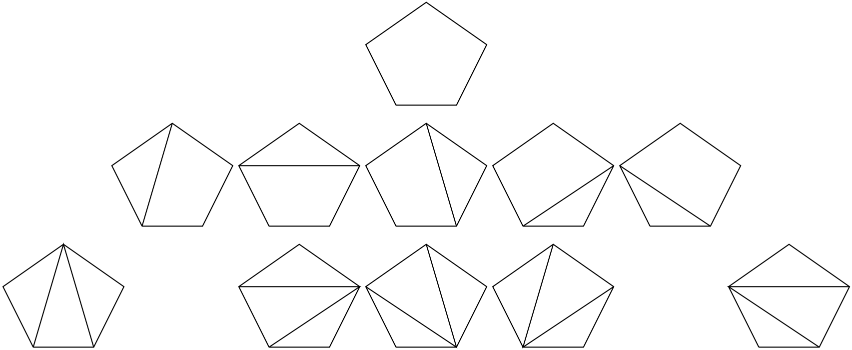
Each dot represents a generator in Khovanov-Rozansky homology, so the total dimension of homology is $5 + 5 + 1 = 11$.

[1] Picture credit: The Knot Atlas

[2] Picture credit: S. Gukov, N. Dunfield, J. Rasmussen

HOMFLY-PT homology: examples

Observe that there are **5** ways to draw two non-intersecting diagonals in a pentagon, **5** ways to draw one diagonal. and **1** way to draw no diagonals, in total $5 + 5 + 1 = 11$.



The number of triangulations of an n-gon is called the **Catalan number**.

HOMFLY-PT homology: examples

The following result was proved by Hogancamp and Mellit in 2017, following my conjecture from 2010:

Theorem (Hogancamp, Mellit)

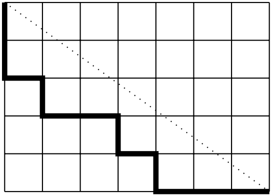
a) The total dimension of HOMFLY homology for the $(n, n+1)$ torus knot equals the number of ways to draw non-intersecting diagonals in the $(n+2)$-gon.
b) The bigraded dimension of the $a = 0$ part of the HOMFLY homology for the $(n, n+1)$ torus knot equals the q, t-Catalan number $c_n(q, t)$.
c) More generally, the bigraded dimension of the $a = 0$ part of the HOMFLY homology for the (m, n) torus knot equals the corresponding rational q, t-Catalan number $c_{m,n}(q, t)$.

HOMFLY-PT homology: examples

The q, t-Catalan numbers and their rational analogues were introduced and studied by Bergeron, Garsia, Haiman, Haglund and many others. They can be defined as

$$c_{m,n}(q,t) = \sum_D q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)}$$

where the sum is over all lattice paths D in the $m \times n$ rectangle which do not cross the diagonal, and $\mathrm{area}(D), \mathrm{dinv}(D)$ are certain combinatorial statistics.



HOMFLY-PT homology: structures

The following result was conjectured by Gukov, Dunfield and Rasmussen in 2005, but took very long time to prove.

Theorem (G., Hogancamp, Mellit, 2021)

The HOMFLY homology of any knot is symmetric around the vertical axis. Furthermore, there is an action of the Lie algebra $\mathfrak{sl}(2)$ in HOMFLY homology which yields this symmetry.

- For torus knots, the symmetry implies $c_{m,n}(q,t) = c_{m,n}(t,q)$, which is a highly nontrivial combinatorial identity.
- The action of $\mathfrak{sl}(2)$ preserves δ-grading and a-grading, so we can look at each row separately
- The representation theory of $\mathfrak{sl}(2)$ implies $\dim \mathcal{H}_{q-4} \le \dim \mathcal{H}_q$ for $q \le 0$ and $\mathcal{H}_q \ge \dim \mathcal{H}_{q+4}$ for $q \ge 0$,so the ranks of HOMFLY homology "grow to the middle" in each row.

HOMFLY-PT homology: structures

What about unreduced homology or links with several components?

For links with several (say, r) components, the unreduced HOMFLY homology is infinite-dimensional and it is a module over the polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$ corresponding to marked points. The symmetry does not preserve the action of x_i.

Hogancamp and I defined a deformation, or y-ification of HOMFLY homology for links which depends on additional variables $y_1, \ldots, y_r$.

Theorem (G., Hogancamp, Mellit, 2021)

The y-ified homology $HY(L)$ of any link is symmetric, and the symmetry exchanges x_i with y_i.

For knots, we have $HY(K) = \mathcal{H}(K) \otimes \mathbb{C}[x, y]$ where $\mathcal{H}(K)$ is the reduced homology, so y-ification does not give any new information.

HOMFLY-PT homology: structures

For example, consider the (n, n) torus link with n unknotted components which are pairwise linked.

Theorem (G., Hogancamp, 2017)

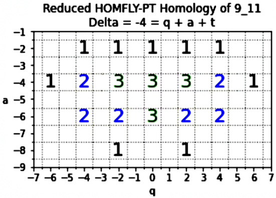
a) The $a = 0$ part of the y-ified homology of the (n, n) torus link is isomorphic to the ideal

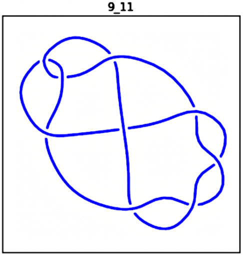
$$J = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j) \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

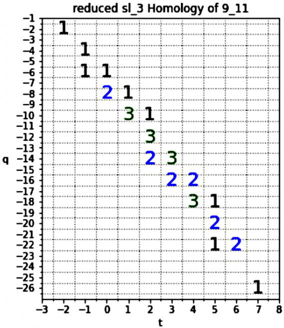
which is the defining ideal for the union of diagonals in $(\mathbb{C}^2)^n$.
b) The $a = 0$ part of the HOMFLY homology of the (n, n) torus link is isomorphic to $J/(y_1, \ldots, y_n)J$.

Theorem (Rasmussen)

For all $N \geq 1$, there is a spectral sequence from (reduced) HOMFLY homology of K to the $\mathfrak{sl}(N)$ homology of K. The q-grading on the latter can be computed as $q + Na$.







HOMFLY-PT homology: structures

Some observations:

- If HOMFLY homology is supported in one δ-grading, then Rasmussen spectral sequences collapse for all $N \geq 2$
- If HOMFLY homology is supported in two δ-gradings, then Rasmussen spectral sequences collapse for all $N \geq 3$
- In particular, we know $\mathfrak{sl}(N)$ homology for $N \geq 3$ for all knots in the Nakagane-Sano dataset
- The spectral sequence for $N = 1$ converges to $\mathfrak{sl}(1)$ homology which is 1-dimensional. Its grading is related to the (generalized) S-invariant
- **Work in progress (G., Chandler):** Can we compute the S invariant for knots in the dataset? What is the relation between HOMFLY and $\mathfrak{sl}(N)$ homology S-invariants?
- Lewark and Lobb have interesting examples where $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ homology S-invariants are different.

HOMFLY-PT homology: geometric models

One of geometric models for HOMFLY homology is given by **braid varieties**. Recall that the braid group on the n strands is defined by generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j|>1).$$

We define matrices

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & z & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

One can check that
$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$.

HOMFLY-PT homology: geometric models

Given a **positive** braid $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, we define the braid variety

$$X(\beta) = \left\{ z_1, \ldots, z_k \ : \ B_{i_1}(z_1) \cdots B_{i_k}(z_k) \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \ \mathrm{upper-triangular} \right\}$$

Theorem (Escobar; Casals, G., M. Gorsky, Simental)

$X(\beta)$ is either empty or it is a smooth manifold of dimension $k - \binom{n}{2}$. If β closes to a knot then $X(\beta) = (\mathbb{C}^*)^{n-1} \times Y(\beta)$ for some $Y(\beta)$.

Theorem (Webster-Williamson, Mellit, Trinh)

*Suppose that β closes to a knot. The $a = 0$ part of the HOMFLY homology is isomorphic to the homology of $Y(\beta)$ equipped with **weight filtration**.*

Example

For $\beta = \sigma_1^3$ we have $Y(\beta) = \{z_1, z_2, z_3 : z_1 + z_3 + z_1 z_2 z_3 = 1\} \subset \mathbb{C}^3$.

Theorem (Galashin, Lam, 2021)

For torus knots, the variety $X(\beta)$ is isomorphic (up to $(\mathbb{C}^*)^{\cdots}$) to the **open positroid variety** $\Pi^{\circ}_{m,n} \subset Gr(m, n+m)$ *defined by the non-vanishing of cyclically consecuitive minors. The homology of $\Pi^{\circ}_{m,n}$ equipped with weight filtration are given by q, t-Catalan numbers $c_{m,n}(q, t)$.*

HOMFLY-PT homology: geometric models

Theorem (Kálmán,Casals-Ng)

The braid variety agrees with the **augmentation variety** *of the corresponding Legendrian knot.*

More precisely:

- To a Legendrian knot K, Chekanov associated a differential graded algebra $\mathcal{A}_K$
- The generators correspond to the crossings, and the differential counts certain holomorphic disks
- The above theorem states that the **algebra of functions** on $X(\beta)$ is isomorphic to $H^0(\mathcal{A}_K)$

Open problem: Can we describe the homology of $X(\beta)$ or $Y(\beta)$ using Chekanov dga?

HOMFLY-PT homology: geometric models

There are two more geometric models for HOMFLY homology that involve more complicated algebraic geometry:

1. For **algebraic knots**, Oblomkov-Rasmussen-Shende conjecture the relation between HOMFLY homology and **affine Springer fibers**. The conjecture is open in general, but proved for torus knots (G.-Mazin-Vazirani, Hogancamp-Mellit).

2. For arbitrary knots, there is a deep relation between HOMFLY homology and the **Hilbert scheme of points on the plane** (G.-Neguț-Rasmussen,G.-Hogancamp,Oblomkov-Rozansky).

More examples, details and references:

E. Gorsky, M. Hogancamp, A. Mellit. Tautological classes and symmetry in Khovanov-Rozansky homology. arXiv:2103.01212

E. Gorsky, O. Kivinen, J. Simental. Algebra and geometry of link homology. arXiv:2108.10356

K. Nakagane, T. Sano. Computations of HOMFLY homology. arXiv:2111.00388

Thank You!