Elliptic Hall algebra and its categorifications

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Diagrammatic Categorification
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The elliptic Hall algebra $\mathcal{E}$ is a remarkable algebra over $\mathbb{Q}(q,t)$ with connections to many areas of mathematics including:

- Geometry of elliptic curves over $\mathbb{F}_q$ (Burban, Schiffmann)
- Combinatorics of Macdonald polynomials/Shuffle Theorem (Bergeron-Garsia-Leven-Xin, Blasiak-Haiman-Morse-Pun-Seelinger,...)
- Geometry of Hilbert schemes and commuting stacks (Schiffmann-Vasserot, Neguț,...)
- Khovanov-Rozansky homology (G.-Neguț, Hogancamp, Mellit...)
- Skein algebras (Morton-Samuelson,...)

In this talk, I will describe several explicit presentations of $\mathcal{E}$ (or rather of the positive half $\mathcal{E}^+$) by generators and relations, and outline some approaches and challenges towards its categorification.

As quantum toroidal algebra

The easiest description of $\mathcal{E}^{>}$ has generators $e_k, k \in \mathbb{Z}$ which are packed into the generating function $e(\mathbf{z}) = \sum e_i \mathbf{z}^{-i}$. These are subject to relations

$$e(\mathbf{z})e(\mathbf{w})\left(1-q\frac{\mathbf{w}}{\mathbf{z}}\right)\left(1-t\frac{\mathbf{w}}{\mathbf{z}}\right)\left(1-qt\frac{\mathbf{z}}{\mathbf{w}}\right) = e(\mathbf{w})e(\mathbf{z})\left(1-q\frac{\mathbf{z}}{\mathbf{w}}\right)\left(1-t\frac{\mathbf{z}}{\mathbf{w}}\right)\left(1-qt\frac{\mathbf{w}}{\mathbf{z}}\right) \quad (1)$$

and

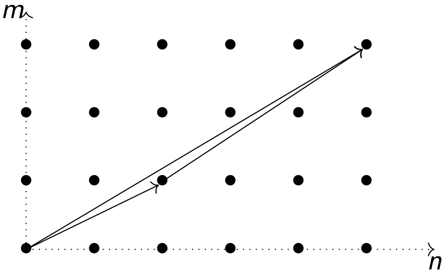
$$[e_0, [e_1, e_{-1}]] = 0.$$

As usual, both sides of the equation (1) should be expanded as Laurent series in $\mathbf{z}, \mathbf{w}$ and for all $(a, b) \in \mathbb{Z}^2$ one compares the coefficients at $\mathbf{z}^{-a}\mathbf{w}^{-b}$. Note that we get 8+8=16 terms for each (a, b), which might be hard to categorify.

As elliptic Hall algebra

The algebra $\mathcal{E}^{>}$ has generators $P_{n,m}$ with $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$, and certain relations. In particular, if $mn' - m'n = 1$, or equivalently there are no lattice points in the triangle Δ with vertices $(0,0), (n,m), (n+n', m+m')$ then

$$[P_{n,m}, P_{n',m'}] = (scalar) P_{n+n', m+m'}.$$

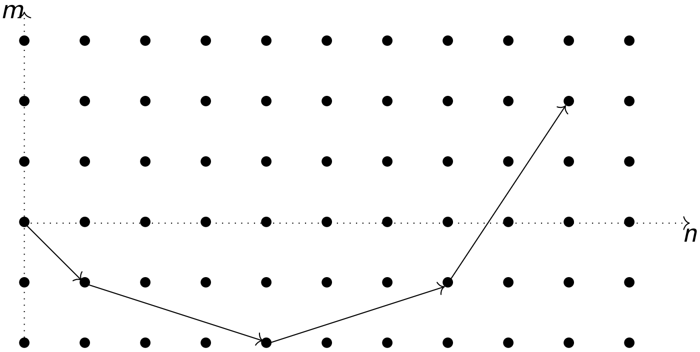


However, if Δ has lattice points inside or on the boundary, the relations are significantly more complicated. We have $P_{1,m} = e_m$. It is quite complicated to write $P_{n,m}$ in terms of e_k.

Burban, Schiffmann: The algebra $\mathcal{E}^{>}$ has a basis given by products

$$P_{n_1,m_1} \cdots P_{n_\ell,m_\ell}, \quad \frac{m_1}{n_1} \leq \cdots \leq \frac{m_\ell}{n_\ell}.$$

In other words, the vectors $(n_1, m_1), \ldots, (n_\ell, m_\ell)$ form a **convex path** γ which passes through the lattice points and the basis is parametrized by all such convex paths:



Unified presentation

Neguț developed a new presentation of $\mathcal{E}^{>}$ with generators $Y_{d_1,\dots,d_\ell}$ and relations

$$Y_{d_1,\dots,d_i,d_{i+1},\dots,d_\ell} - qtY_{d_1,\dots,d_i+1,d_{i+1}-1,\dots,d_\ell} = (q-1)\,Y_{d_1,\dots,d_i}\,Y_{d_{i+1},\dots,d_\ell}$$

$$[Y_k, Y_{d_1,\dots,d_n}] = (t-1)(q-1)\sum_{i=1}^{n}\begin{cases}\sum_{a=1}^{k-d_i} Y_{d_1,\dots,d_{i-1},k-a,d_i+a,d_{i+1},\dots,d_n} \text{ if } k > d_i \\ 0 \text{ if } k = d_i \\ -\sum_{a=1}^{d_i-k} Y_{d_1,\dots,d_{i-1},d_i-a,k+a,d_{i+1},\dots,d_n} \text{ if } k < d_i\end{cases}$$

These generators include all of the above since

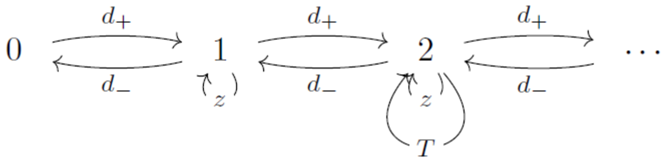
$$e_k = Y_k,\;\; P_{n,m} = Y_{S_1(m,n),\dots,S_n(m,n)},\;\text{ where } S_i(m,n) = \left\lfloor \frac{im}{n} \right\rfloor - \left\lfloor \frac{(i-1)m}{n} \right\rfloor.$$

As subalgebra of $\mathbb{B}_{q,t}$

The algebra $\mathbb{B}_{q,t}$ was introduced by Carlsson and Mellit in their proof of Shuffle Conjecture, it has orthogonal idempotents $\mathbf{1}_k,\ k \geq 0$, generators

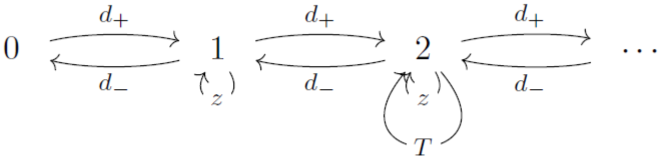
$$\mathbf{1}_k z_i \mathbf{1}_k\ (1 \leq i \leq k),\ \mathbf{1}_k T_i \mathbf{1}_k\ (1 \leq i \leq k-1),\ \mathbf{1}_{k+1} d_+ \mathbf{1}_k,\ \mathbf{1}_{k-1} d_- \mathbf{1}_k$$

and some relations. Schematically, the generators can be presented by the following quiver:



In particular, for each k the elements $\mathbf{1}_k z_i \mathbf{1}_k, \mathbf{1}_k T_i \mathbf{1}_k$ generate a copy of the affine Hecke algebra AH_k.

As subalgebra of $\mathbb{B}_{q,t}$



Theorem (Gonzalez,G., Simental)

We have the isomorphism of algebras $\mathcal{E}^{>} \simeq \mathbf{1}_0\mathbb{B}_{q,t}\mathbf{1}_0$. In particular, under this isomorphism we can identify

$$e_r = \mathbf{1}_0 d_- z_1^r d_+ \mathbf{1}_0, \ \ Y_{d_1,\dots,d_\ell} = \mathbf{1}_0 d_- z_1^{d_1} \varphi z_1^{d_2} \cdots \varphi z_1^{d_\ell} d_+ \mathbf{1}_0.$$

where $\varphi = \frac{1}{q-1}[d_+, d_-]$.

As K-theoretic Hall algebra

The **commuting stack** Comm_n is defined as the quotient

$$\mathrm{Comm}_n = \{X, Y \in \mathrm{Mat}_n : [X, Y] = 0\}/\mathrm{GL}_n.$$

Often we will implicitly require that Y is nilpotent. It has an action of the two-dimensional torus $T = \mathbb{C}^\times \times \mathbb{C}^\times$ by scaling X and Y, and we will be interested in the equivariant K-theory. One can define the convolution products

$$K_T(\mathrm{Comm}_n) \otimes K_T(\mathrm{Comm}_m) \xrightarrow{*} K_T(\mathrm{Comm}_{m+n})$$

so $\bigoplus_{n=0}^{\infty} K_T(\mathrm{Comm}_n)$ has an algebra structure.

Theorem (Schiffmann,Varagnolo,Vasserot)

There is an isomorphism of algebras

$$\bigoplus_{n=0}^{\infty} K_T(\mathrm{Comm}_n) \simeq \mathcal{E}^{>}.$$

Geometric categorification

One can consider the T-equivariant derived category of the commuting stack $D_T(\mathrm{Comm}_n)$ as a geometric categorification of $\mathcal{E}^{>}$. But what does it mean concretely (for *diagrammatic categorification*)?

Geometric categorification

One can consider the T-equivariant derived category of the commuting stack $D_T(\mathrm{Comm}_n)$ as a geometric categorification of $\mathcal{E}^>$. But what does it mean concretely (for *diagrammatic categorification*)?
Negut defined the explicit objects $Y_{d_1,\dots,d_\ell} \in D_T(\mathrm{Comm}_\ell)$ corresponding to the namesake generators of $\mathcal{E}^>$. Furthermore, the relations get categoried:

Theorem (Neguț, Zhao)

For all integers $d_1, \dots, d_\ell$ and all $i \in \{1, \dots, \ell - 1\}$, there is a morphism:

$$Y_{d_1,\dots,d_i,d_{i+1},\dots,d_\ell} \to Y_{d_1,\dots,d_i-1,d_{i+1}+1,\dots,d_\ell}$$

with the cone filtered by two copies of $Y_{d_1,\dots,d_i} * Y_{d_{i+1},\dots,d_\ell}$.

Theorem (Neguț, Zhao)

Given $d_1, \ldots, d_\ell \in \mathbb{Z}$, there are explicit objects $G_0, \ldots, G_\ell \in D_T(\mathrm{Comm}_{\ell+1})$ such that

(1) $G_0 = Y_{d_1,\ldots,d_\ell} * Y_k$ *and* $G_\ell = Y_k * Y_{d_1,\ldots,d_\ell}$

(2) *for all $i \in \{1, \ldots, \ell\}$, there exist explicit morphisms*

$$\begin{cases} G_{i-1} \to G_i & \text{if } d_i > k \\ G_{i-1} \leftarrow G_i & \text{if } d_i < k \\ G_{i-1} \simeq G_i & \text{if } d_i = k. \end{cases}$$

(3) *for all $i \in \{1, \ldots, \ell\}$, the cone of the morphism in the previous item has a filtration with the associated graded object given by*

$$K \otimes \begin{cases} \bigoplus_{a=k}^{d_i-1} Y_{d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots,d_\ell} & \text{if } d_i > k \\ \bigoplus_{a=d_i}^{k_i-1} Y_{d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots,d_\ell} & \text{if } d_i < k. \end{cases}$$

Her K is an explicit Koszul complex.

Cautis, Pădurariu and Toda initiated an ambitious program categorifying the basis of convex paths. In short, they develop a two-step decomposition of the derived category $D_T(\mathrm{Comm}_n)$. At the first step, they consider the **orthogonal decomposition**:

$$D_T(\mathrm{Comm}_n) = \bigoplus_{m \in \mathbb{Z}} D_T(\mathrm{Comm}_n; m) \tag{2}$$

categorifying the bigrading on $\mathcal{E}^{>}$. There are no nonzero morphisms between different blocks in (2). At the second step, they consider the decomposition for each block:

$$D_T(\mathrm{Comm}_n; m) = \langle \mathcal{C}_\gamma \ : \ \gamma = \text{convex path from } (0,0) \text{ to } (n,m) \rangle\,. \tag{3}$$

The subcategory $\mathcal{C}_\gamma$ is referred to as "quasi-BPS category" for γ. Remarkably, (3) is a **semiorthogonal decomposition**: a nonzero morphism from an object in $\mathcal{C}_\gamma$ to another object in $\mathcal{C}_{\gamma'}$ is only possible if the path γ is (non-strictly) above the path γ'.

Problem: Describe the subcategories $\mathcal{C}_\gamma$ explicitly.

Affine Soergel bimodules

Let $R = \mathbb{C}[x_1, \dots, x_n, \delta]$. We have an endomorphism of R given by:

$$\pi(\delta) = \delta, \pi(x_n) = x_1 - \delta, \pi(x_i) = x_{i+1}, 1 \leq i \leq n-1.$$

We consider $R - R$ bimodules $B_i := R \otimes_{R^{s_i}} R$ and additional bimodule Ω which is isomorphic to R where the left action is standard and the right action is twisted by π. The category ASBim_n is the smallest subcategory of $R - R$ bimodules containing R, Ω and B_i and closed under grading shifts, tensor products, direct sums and direct summands.

Theorem (Soergel+Elias, Mackaay-Thiel)

ASBim_n *categorifies the affine Hecke algebra:* $K_0(\mathrm{ASBim}_n) = \mathrm{AH}_n$.

Roquier complexes

Define the following complexes: $T_i := [B_i \to R]$, $T_i^{-1} := [R \to B_i]$.

Theorem (Rouquier, Elias)

The complexes T_i, T_i^{-1} and Ω satisfy affine braid relations and thus categorify the affine braid group.

In particular, we can define

$$\mathcal{L}_i = T_{i-1}^{-1} \cdots T_1^{-1} \Omega T_{n-1} \cdots T_i, \ \mathcal{L}_i \mathcal{L}_j = \mathcal{L}_j \mathcal{L}_i.$$

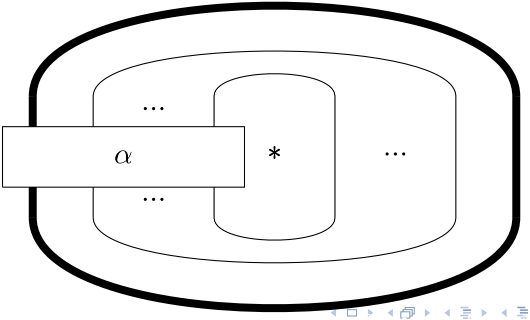
Let $\mathcal{C}$ be a rigid monoidal (dg) category. With Hogancamp and Wedrich, we defined another dg category $\mathrm{Tr}(\mathcal{C})$ with the following properties:

(a) There is a trace functor $\mathcal{C} \to \mathrm{Tr}(\mathcal{C})$

(b) There is an isomorphism $\mathrm{Tr}(XY) \simeq \mathrm{Tr}(YX)$ for all $X, Y \in \mathcal{C}$.

(c) We have $\mathrm{End}(\mathrm{Tr}(\mathbf{1})) \simeq \mathrm{HH}_*(\mathcal{C})$, the Hochschild homology of the category $\mathcal{C}$.

Informally, one can think of objects in $\mathrm{Tr}(\mathcal{C})$ as of closures of objects in $\mathcal{C}$ in the annulus. In particular, one can think of $\mathrm{Tr}(\mathrm{ASBim}_n)$ as the categorification of the HOMFLY skein algebra of the torus.

$\mathrm{Tr}(\alpha) =$ 

Given a vector $(d_1, \ldots, d_n)$, one can define the object

$$\mathcal{Y}_{d_1,\ldots,d_n} := \mathrm{Tr}\left(\mathcal{L}_1^{d_1} \cdots \mathcal{L}_n^{d_n} T_1 \cdots T_{n-1}\right) \in \mathrm{Tr}(\mathrm{ASBim}_n),$$

In particular, for $\mathrm{GCD}(m,n) = 1$ and $d_i = S_i(m,n)$ we get

$$\mathcal{P}_{n,m} := \mathcal{Y}_{S_1(m,n),\ldots,S_n(m,n)} := \mathrm{Tr}(\Omega^m) \in \mathrm{Tr}(\mathrm{ASBim}_n).$$

Furthermore, for a sequence $i_1 < \ldots < i_k$ we define

$$\mathcal{Y}_{d_1,\ldots,d_{i_1-1}} * \mathcal{Y}_{d_{i_1},\ldots,d_{i_2-1}} * \cdots \mathcal{Y}_{d_{i_k},\ldots,d_n} = \mathrm{Tr}\left(\mathcal{L}_1^{d_1} \cdots \mathcal{L}_n^{d_n} T_1 \cdots \widehat{T_{i_1}} \cdots \widehat{T_{i_k}} \cdots T_{n-1}\right).$$

Theorem (G.,Neguț)

a) The Karoubi completion of the dg category $\mathrm{Tr}(\mathrm{ASBim}_n)$ *is generated by the direct summands of the products of* $\mathcal{Y}_{\underline{d}}$ *which satisfy the analogues of the exact sequences in geometric category.*

b) The Karoubi completion of the dg category $\mathrm{Tr}(\mathrm{ASBim}_n)$ *is generated by the direct summands of the objects* $\mathcal{P}_{n_1,m_1} * \cdots * \mathcal{P}_{n_\ell,m_\ell}$ *with* $n_1 + \ldots + n_\ell = n$ *and* $\frac{m_1}{n_1} \leq \cdots \leq \frac{m_\ell}{n_\ell}$.

c) For $\mathrm{GCD}(m,n) = 1$ *we have*

$$\mathrm{End}(\mathcal{P}_{n,m}^{*d}) \simeq \mathbb{C}[x_1,\ldots,x_d,\theta_1,\ldots,\theta_d] \rtimes \widetilde{S_d}$$

where $\widetilde{S_d}$ *is the affine symmetric group.*

Theorem (G. Neguț)

The category $\mathrm{Tr}(\mathrm{ASBim}_n)$ *has an orthogonal decomposition*

$$\mathrm{Tr}(\mathrm{ASBim}_n) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Tr}(\mathrm{ASBim}_n; m).$$

There are no nonzero morphisms between different blocks of this decomposition.

Problem

Is there a semiorthogonal decomposition of $\mathrm{Tr}(\mathrm{ASBim}_n; m)$ *indexed by convex paths* γ*? How to describe the subcategory* $\widetilde{\mathcal{C}}_\gamma$ *for a given* γ*?*

The convex paths γ turn out to be in bijection with minimal length representatives in conjugacy classes in the affine symmetric group $\widetilde{S}_n$.

If γ is a straight line we can describe the subcategory $\widetilde{\mathcal{C}}_\gamma$ explicitly.

Theorem (G., Neguț)

Assume that γ is a straight line from $(0,0)$ to (n,m) with $\mathrm{GCD}(n,m)=d$. Then the subcategory $\widetilde{\mathcal{C}}_\gamma$ is generated by the direct summands of the single object $\mathcal{P}^{*d}_{\frac{n}{d},\frac{m}{d}}$. Furthermore, the endomorphism algebra of $\mathcal{P}^{*d}_{\frac{n}{d},\frac{m}{d}}$ is given by $\mathbb{C}[x_1,\ldots,x_d,\theta_1,\ldots,\theta_d] \rtimes \widetilde{S_d}$ and its indecomposable summands are indexed by partitions of d. The Grothendieck group $K_0\left(\widetilde{\mathcal{C}}_\gamma\right)$ is isomorphic to the space of degree d symmetric functions in infinitely many variables.

Given $(n_1,m_1),\ldots,(n_k,m_k)$ with $n_1+\ldots,n_k=n, m_1+\ldots+m_k=m$ and $\frac{m_1}{n_1}=\cdots=\frac{m_k}{n_k}$, the product $\mathcal{P}_{n_1,m_1} * \cdots \mathcal{P}_{n_k,m_k}$ belongs to the subcategory $\widetilde{\mathcal{C}}_\gamma$, and we can write an explicit resolution for $\mathcal{P}_{n_1,m_1} * \cdots \mathcal{P}_{n_k,m_k}$ in terms of direct summands of $\mathcal{P}^{*d}_{\frac{n}{d},\frac{m}{d}}$.

Comparing the categorifications

Motivated by the similarity between the two categories, we propose the following:

Conjecture (G., Neguț)

There exist dg functor $\mathrm{Tr}(\mathrm{ASBim}_n) \to D_T(\mathrm{Comm}_n)$ *sending objects* $\mathcal{Y}_{d_1,\ldots,d_n} \in \mathrm{Tr}(\mathrm{ASBim}_n)$ *to the namesake objects* $Y_{d_1,\ldots,d_n} \in D_T(\mathrm{Comm}_n)$.

This would categorify the following result:

Theorem (Morton, Samuelson)

The HOMFLY skein algebra of the torus is isomorphic to the elliptic Hall algebra at $t = q^{-1}$.

Thank you!