Jet spaces in link homology

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Jet schemes

Let X be an affine scheme defined in $\mathbb{C}^A$ by some equations

$$F_1(z_1, \dots, z_A) = \dots = F_B(z_1, \dots, z_A) = 0.$$

Recall that the n-th **jet scheme** $\mathrm{Jet}^n X$ is defined as

$$\mathrm{Spec}\, \frac{\mathbb{C}[z_i^{(j)} : 1 \le i \le A, 0 \le j \le n]}{F_k(z_1(t), \dots, z_A(t)) = 0 \ mod \ t^{n+1},\ 1 \le k \le B}$$

where $z_i(t) = z_i^{(0)} + z_i^{(1)} t + \dots + z_i^{(n)} t^n$. The **reduced** jet scheme plays an important role in motivic integration, and is controlled by the singularities of X.

Jet schemes

In particular, if X is smooth then $\mathrm{Jet}^n X$ is a rank $(n+1)$ vector bundle over X and $\dim \mathrm{Jet}^n X = (n+1) \dim X$. The work of de Fernex, Ein, Lasarsfeld, Mustata et al. related the invariants of singularities of X to those of the jet schemes.

The work of Bruschek, Mourtada and Schepers initiated the study of the **non-reduced** structure of $\mathrm{Jet}^n X$, in particular, they studied the Hilbert series of the corresponding rings of functions.

In this talk, I will descrive some results and conjectures in link homology which suggest that there is a "derived" generalization of the above constructions.

Jet schemes

For simplicity, assume that X is a complete interesection, consider the Koszul complex:

$$\mathcal{K} = (\mathbb{C}[z_1, \ldots, z_A, \xi_1, \ldots, \xi_B], d), \; d(\xi_k) = F_k(z_1, \ldots, z_A), \; d(z_i) = 0.$$

By our assumptions, $H^0(\mathcal{K}) = \mathbb{C}[X]$ and all higher homology vanish.

Definition

Define

$$\mathrm{Jet}^n\mathcal{K} = (\mathbb{C}[z_i^{(j)}, \xi_k^{(j)}], d), \; 1 \leq i \leq A, 1 \leq k \leq B, 0 \leq j \leq n,$$

$$d(\xi_k(t)) = F_k(z_1(t), \ldots, z_A(t)) \mod t^{n+1}, \; d(z_i(t)) = 0.$$

Problem

Compute the homology of $\mathrm{Jet}^n\mathcal{K}$.

Jet schemes

The following will be our motivating example.

Example

Let $X = \mathbb{C}[z]/(z^N)$ and $n = 1$. We have two even variables $z^{(0)}, z^{(1)}$ and two odd variables $\xi^{(0)}, \xi^{(1)}$ with

$$d(\xi^{(0)} + \xi^{(1)} t) = (z^{(0)} + z^{(1)} t)^N \mod t^2.$$

That is,

$$d(\xi^{(0)}) = (z^{(0)})^N, \; d(\xi^{(1)}) = N(z^{(0)})^{N-1} z^{(1)}.$$

Let $\mu = Nz^{(1)}\xi^{(0)} - z^{(0)}\xi^{(1)}$, then it is easy to see that

$$d(\mu) = 0, \; d(\xi^{(0)}\xi^{(1)}) = -(z^{(0)})^{N-1}\mu.$$

The homology of $\mathrm{Jet}^1\mathcal{K}$ is generated by $z^{(0)}, z^{(1)}$ and μ modulo relations

$$(z^{(0)})^N = N(z^{(0)})^{N-1} z^{(1)} = (z^{(0)})^{N-1}\mu = 0.$$

Example

We have $H^0 \mathrm{Jet}^n \mathcal{K} = \mathbb{C}[\mathrm{Jet}^n X]$.

Higher homology $H^i \mathrm{Jet}^n \mathcal{K}$ are modules over $H^0 \mathrm{Jet}^n \mathcal{K}$, and hence correspond to some sheaves on $\mathrm{Jet}^n X$. It would be interesting to know if these sheaves carry some geometric information about $\mathrm{Jet}^n X$ or X.

Now we take a digression to discuss link invariants. As we will see, the above example of $\mathrm{Jet}^1 \mathcal{K}$ corresponds to the $\mathfrak{gl}(N)$ **Khovanov-Rozansky homology** of two-strand torus knots.

Link invariants

Given a semisimple Lie algebra $\mathfrak{g}$ and a representation V of the corresponding quantum group $U_q\mathfrak{g}$, one can define **Reshetikhin-Turaev link invariants**. To any link L in $\mathbb{R}^3$, this assigns a polynomial $P_{\mathfrak{g},V}(L;q)$ which depends on a single variable q. Some examples include:

- For $\mathfrak{g} = \mathfrak{gl}(2)$ (or $\mathfrak{g} = \mathfrak{sl}(2)$) and $V = \mathbb{C}^2$, one gets the **Jones polynomial**.
- For $\mathfrak{g} = \mathfrak{gl}(2)$ and $V = S^k\mathbb{C}^2$, one gets the colored Jones polynomial.
- For $\mathfrak{g} = \mathfrak{gl}(N)$ and $V = \mathbb{C}^N$, the polynomial can be computed recursively using **skein relation**:

$$q^{-N}P(\nwarrow\!\!\!\!\nearrow) - q^{N}P(\nwarrow\!\!\!\!\nearrow) = (q^{-1} - q)P(\uparrow\,\uparrow)$$

- For $\mathfrak{g} = \mathfrak{gl}(N)$ and $V = \wedge^k\mathbb{C}^N$, there are more complicated recursions due to Murakami-Ohtsuki-Yamada (MOY), reinterpreted via **web diagrams** of Cautis-Kamnitzer-Morrison.

Link invariants

Some basic properties of Reshetikhin-Turaev invariants:

- The invariant of the unknot is given by the q-character of V.
- In particular, for $\mathfrak{g} = \mathfrak{gl}(N)$ and $V = \mathbb{C}^N$ the invariant of the unknot equals (up to normalization)

$$P(O;q) = \frac{1-q^N}{1-q} = 1 + q + \ldots + \ldots + q^{N-1}$$

- For $\mathfrak{g} = \mathfrak{gl}(N)$ and any V, the invariants of torus knots $T(m,n)$ are known (Rosso-Jones)
- Given a Young diagram λ, there exists a **colored HOMFLY-PT** link invariant $P_\lambda(L;a,q)$ such that

$$P_\lambda(L; a = q^N, q) = P_{\mathfrak{gl}(N), V_\lambda}(L;q),$$

  where V_λ is the irreducible representation of $\mathfrak{gl}(N)$ labeled by λ.
- For example, for the unlink and $\lambda = \square$ we get

$$P(O;a,q) = \frac{1-a}{1-q} \xrightarrow{a=q^N} \frac{1-q^N}{1-q}.$$

Link invariants

In recent decades, Khovanov, Rozansky and their collaborators developed the idea of **link homology** which **categorify** Reshetikhin-Turaev invariants:

- For $\mathfrak{g} = \mathfrak{gl}(2)$ and $V = \mathbb{C}^2$, the original **Khovanov** homology is a bigraded vector space $\mathrm{Kh}(L) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(L)$ such that its graded Euler characteristic $\sum (-1)^i q^j \dim \mathrm{Kh}^{i,j}(L)$ recovers the Jones polynomial.
- Khovanov and Rozansky defined **gl**(N) homology $\mathcal{H}_N(L)$ whose Euler characteristics recovers $(\mathfrak{gl}(N), \mathbb{C}^N)$ link invariant.
- Separately, Khovanov and Rozansky defined triply graded HOMFLY-PT homology $\mathrm{HHH}(L)$ whose Euler characteristics recovers HOMFLY-PT link invariant for $\lambda = \square$.

Lots of other constructions (Cautis-Kamnitzer, Queffelec-Rose, Robert-Wagner, Webster-Williamson...) generalize this to other representations V and other $\mathfrak{g}$. In particular, one can define HOMFLY-PT homology for arbitrary color λ.

Link invariants

Some basic properties of link homology:

- The $\mathbf{gl}(N)$ homology of the unknot is a graded algebra.
- For $(\mathfrak{gl}(N), \mathbb{C}^N)$ we get $H^*(\mathbb{CP}^{N-1}) = \mathbb{C}[x]/(x^N)$.
- For $(\mathfrak{gl}(N), \wedge^k \mathbb{C}^N)$ we get $H^*(\mathrm{Gr}(k, N)) = \mathbb{C}[e_1, \ldots, e_k]/(f_1, \ldots, f_k)$.

Theorem (Rasmussen)

For each N and anly link L, there is a spectral sequence from $\mathrm{HHH}(L)$ to $\mathcal{H}_N(L)$. In many cases there is only one nontrivial differential d_N such that

$$H^*(\mathrm{HHH}(L), d_N) = \mathcal{H}_N(L).$$

Example

For example, for the unknot and $\lambda = \square$ we get $\mathrm{HHH}(O) = \mathbb{C}[x, \xi]$ and $d_N(\xi) = x^N$. The homology of d_N is precisely $\mathbb{C}[x]/(x^N)$.

Example

For example, for the unknot and $\lambda = \wedge^k$ we get $\mathrm{HHH}_{\wedge^k}(O) = \mathbb{C}[e_1, \ldots, e_k, \xi_1, \ldots, \xi_k]$ and

$$d_N(e_i) = 0, \ d_N(\xi_i) = f_i(e_1, \ldots, e_k)$$

where f_i are the defining equations of $H^*(\mathrm{Gr}(k, N))$. In other words, d_N defines a **Koszul complex** and here we use the fact that $H^*(\mathrm{Gr}(k, N))$ is a zero-dimensional **complete intersection**.

Main problem

Problem

Open problem: *Compute Khovanov (Khovanov-Rozansky...) homology of torus knots* $T(n,m)$.

Theorem (Stošić)

There is a well defined limit $\lim_{m\to\infty} \mathrm{Kh}(T(n,m))$, *denoted by* $\mathrm{Kh}(T(n,\infty))$.

Problem

Easier (?) open problem: *Compute stable Khovanov (Khovanov-Rozansky...) homology of* $T(n,\infty)$.

Main conjecture

The HOMFLY-PT homology of torus knots $T(n,m)$ is known due to the work of Elias, Hogancamp, Mellit and others. In particular:

Theorem (Hogancamp)

Stable HOMFLY-PT homology of $T(n,\infty)$ is isomorphic to

$$\mathrm{HHH}(T(n,\infty)) = \mathbb{C}[x_0, x_1, \ldots, x_{n-1}, \xi_0, \ldots, \xi_{n-1}].$$

Conjecture (G.,Oblomkov,Rasmussen)

Stable Khovanov homology of $T(n,\infty)$ is isomorphic to

$$\mathrm{Kh}(T(n,\infty)) = H^*(\mathrm{HHH}(T(n,\infty)), d_2), \quad d_2(\xi_k) = \sum_{i+j=k} x_i x_j,\ d_2(x_i) = 0.$$

Main conjecture cont'd

Consider the generating series

$$x(t) = x_0 + x_1 t + \ldots + x_{n-1} t^{n-1}, \quad \xi(t) = \xi_0 + \xi_1 t + \ldots + \xi_{n-1} t^{n-1}$$

Then we can rephrase the conjecture as follows:

Conjecture (G.,Oblomkov,Rasmussen)

Stable Khovanov homology of $T(n, \infty)$ is isomorphic to

$$\mathrm{Kh}(T(n,\infty)) = H^*(\mathbb{C}[x_0, x_1, \ldots, x_{n-1}, \xi_0, \ldots, \xi_{n-1}], d_2),$$

$$d_2(\xi(t)) = x(t)^2 \mod t^n, \ d_2(x(t)) = 0.$$

The conjecture is proved for $n \leq 3$ and agrees with all known data of Khovanov homology up to $n \leq 8$.

Example

For $n = 2$ we have two even variables x_0, x_1 and two odd variables ξ_0, ξ_1 with

$$d_2(\xi_0) = x_0^2,\ d_2(\xi_1) = 2x_0x_1.$$

Note that $d_2(\mu_0) = 0$ where $\mu_0 = 2x_1\xi_0 - x_0\xi_1$, so this Koszul complex has higher homology. At the same time, $x_0\mu_0 = d_2(\xi_0\xi_1)$. We get:

$$H^0(T(2,\infty)) = \mathbb{C}[x_0, x_1]/(x_0^2, 2x_0x_1),\ H^1(T(2,\infty)) = \mathbb{C}[x_0, x_1]\langle\mu_0\rangle/(x_0\mu_0).$$

The complex is bigraded as follows:

$$\deg(x_0) = q^2,\ \deg(x_1) = q^4t^2,\ \deg(\xi_0) = q^4t,\ \deg(\xi_1) = q^6t^3.$$

The differential preserves the q-degree and decreases the t-degree by 1. The Hilbert series of the homology equals:

$$H^0 : q^2 + \frac{1}{1 - q^4t^2},\ H^1 : \frac{q^8t^3}{1 - q^4t^2}.$$

H^0

In general we get

$$H^0 = \mathbb{C}[x_0, \ldots, x_{n-1}]/(x(t)^2 = 0 \mod t^n).$$

The is the ring of functions at the $(n-1)$-st jet scheme $\mathrm{Jet}^{n-1}\mathrm{Spec}\mathbb{C}[x]/(x^2)$ considered by Bruschek, Mourtada and Schepers. At $n = \infty$ it also agrees with the "principal subspace" of a certain $A_1^{(1)}$ module defined by Capparelli-Lepowski-Milas et al, and Feigin-Stoyanovsky, and its Hilbert series is related to the Rogers-Ramanujan identity.

Theorem (Bai,G., Kivinen)

1) The Hilbert series $H_n^0 = H^0(T(n, \infty))$ is given by the recursion

$$H_n^0(Q, T) = \frac{H_{n-2}^0(Q, QT) + tH_{n-3}^0(Q, Q^2T)}{1 - Q^{n-1}T}.$$

2) The projective dimension of H_n^0 equals $\lceil \frac{2n}{3} \rceil$ while the dimension of $\mathrm{Jet}^{n-1}\mathrm{Spec}\mathbb{C}[x]/(x^2)$ equals $\lceil \frac{n-1}{2} \rceil$.

We also have two closed formulas for H^0, comparing these leads to a finite version of the Rogers-Ramanujan identity:

Theorem (Bai,G., Kivinen)

a) We have

$$H_n^0(Q,T) = \sum_{p=0}^{\infty} \frac{\binom{h(n,p)+1}{p}_Q \cdot Q^{p(p-1)} T^p}{(1-Q^{n-h(n,p)}T)\cdots(1-Q^{n-1}T)}$$

where $h(n,p) = \lfloor \frac{n-p}{2} \rfloor$.

b) We have

$$H_n^0(Q,T) = \frac{1}{\prod_{i=0}^{n-1}(1-Q^iT)} \sum_{p=0}^{\infty} (-1)^p \prod_{k=0}^{p-1}(1-Q^kT) \times$$

$$\left(Q^{\frac{5p^2-3p}{2}} T^{2p} \binom{n-2p+1}{p}_Q - Q^{\frac{5p^2+5p}{2}} T^{2p+2} \binom{n-2p-1}{p}_Q \right)$$

Here Q, T are related to q, t by a monomial change of variables.

Higher homology

We have $d(\xi(t)) = x(t)^2, \; d(\dot{\xi}(t)) = 2x(t)\dot{x}(t)$, so

$$d(\mu(t)) = 0, \; \mu(t) = 2\dot{x}(t)\xi(t) - x(t)\dot{\xi}(t) = \mu_0 + \ldots + \mu_{n-1}t^{n-1} \mod t^{n-1}.$$

Theorem (Bai,G.,Kivinen)

The syzygys between $d_2(\xi_i)$ (in other words, the first homology H^1) is generated by μ_i over $\mathbb{C}[x_0, \ldots, x_{n-1}]$.

Conjecture (G., Oblomkov, Rasmussen)

The homology of d_2 is generated (as an algebra) by x_i and μ_i modulo relations

$$x(t)^2 = 0, \; x(t)\mu(t) = 0, \; \ddot{x}(t)\mu(t) - \dot{x}(t)\dot{\mu}(t) = 0, \; \mu(t)\dot{\mu}(t) = 0.$$

We also have a precise, yet conjectural formulas for the Hilbert series of the homology of d_2.

More conjectures

Conjecture (G.,Oblomkov,Rasmussen)

Stable $\mathfrak{gl}(N)$ homology of $T(n,\infty)$ is isomorphic to

$$\mathcal{H}_N(T(n,\infty)) = H^*(\mathbb{C}[x_0, x_1, \ldots, x_{n-1}, \xi_0, \ldots, \xi_{n-1}], d_N),$$

$$d_N(\xi(t)) = x(t)^N \mod t^n,\ d_N(x(t)) = 0.$$

For $N = 3$ this was extensively checked agains link homology data by G.-Lewark. At level zero, we get $\mathrm{Jet}^{n-1}\mathrm{Spec}\mathbb{C}[x]/(x^N)$ which is related to "higher level" variants of Rogers-Ramanujan identity at $n = \infty$. There is also an analogue of $\mu(t)$ given by

$$\mu_N(t) = N\dot{x}(t)\xi(t) - x(t)\dot{\xi}(t),\ d_N(\mu_N(t)) = 0.$$

More conjectures cont'd

Conjecture (G.,Gukov,Stošić)

Suppose that $H^*(\mathrm{Gr}(k,N)) = \mathbb{C}[e_1,\ldots,e_k]/(f_1,\ldots,f_k)$. *Then stable* $\wedge^k$*-colored* $\mathfrak{gl}(N)$ *homology of* $T(n,\infty)$ *is isomorphic to*

$$\mathcal{H}_{N,\wedge^k}(T(n,\infty)) = H^*(\mathbb{C}[e_1(t),\ldots,e_k(t),\xi_1(t),\ldots,\xi_k(t)],d_N),$$

$$d_N(\xi_i(t)) = f_i(e_1(t),\ldots,e_k(t)) \mod t^n,\ d_N(e_i(t)) = 0.$$

Theorem (J. Wang, in progress)

Conjecture is true for $n=2$ *(that is,* $T(2,\infty)$*) and arbitrary* N *and* k.

Deformations

Khovanov homology has several deformations which are important in knot theory. The easiest is so-called **equivariant Khovanov homology** which assigns to the unknot

$$\mathrm{Kh}_{\mathrm{eq}}(O) = \mathbb{C}[x]/(x^2 - bx - c).$$

Here b and c are formal parameters.

Conjecture

Stable equivariant Khovanov homology of $T(n,\infty)$ is isomorphic to

$$\mathrm{Kh}_{eq}(T(n,\infty)) = H^*(\mathbb{C}[x_0, x_1, \ldots, x_{n-1}, \xi_0, \ldots, \xi_{n-1}], d_{2,\mathrm{eq}}),$$

$$d_{2,\mathrm{eq}}(\xi(t)) = x(t)^2 - bx(t) - c \mod t^n,\ d_{2,\mathrm{eq}}(x(t)) = 0.$$

There are also more subtle deformations such as "y-ification" (G.-Hogancamp) or Batson-Seed homology.

Questions

- Is there a VOA interpretation of higher homology of d_N?
- Are there recursions/closed formulas for higher homology of d_N?
- Is there a VOA interpretatoon of the equivariant Khovanov homology?
- There is a lot of torsion in the homology of d_N. Is it possible to use representation theory to control or predict the torsion?
- There is a growing list of homological operations in link homology, in particular, Witt algebra action of Khovanov-Rozansky and "tautological classes" of G.-Hogancamp-Mellit. Are these related to the VOA action?
- Is there some topological interepretation of the recursions?

Thank you!