On Stable Khovanov Homology of Torus Knots (joint with A. Oblomkov, J. Rasmussen)

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Consider the Khovanov homology of the (n, m) torus knot $T(n, m)$

Theorem (M. Stosic)

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Theorem (M. Stosic)

There exists a limit $\mathrm{Kh}(n, \infty) = \lim_{m\to\infty} \mathrm{Kh}(T(n, m))$.

Consider the space $\mathcal{H}_n = \mathbb{Z}[x_0, \ldots, x_{n-1}, \xi_0, \ldots, \xi_{n-1}]$. The variables x_i are even, the variables ξ_i are odd and

$$\deg(x_i) = q^{2i+2}t^{2i}, \quad \deg(\xi_i) = q^{2i+4}t^{2i+1}.$$

Conjecture (G., A. Oblomkov, J. Rasmussen)

The stable Khovanov homology of torus knots can be computed as the homology of the following Koszul complex

$$\mathrm{Kh}(n, \infty) = H^*(\mathcal{H}_n, d_2), \quad d_2(\xi_i) = \sum_{j=0}^{i} x_j x_{i-j}.$$

We have $\mathcal{H}_2 = \mathbb{Z}[x_0, x_1, \xi_0, \xi_1]$, and the differential is given by

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$\mathbb{Q}$-homology is generated by x_0, x_1 and $\mu_0 = 2x_1\xi_0 - x_0\xi_1$ modulo relations $x_0^2 = 2x_0x_1 = x_0\mu_0 = 0$.

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$$\mathrm{Kh}(2, \infty, \mathbb{Q}) = \langle 1, x_0, x_1, \mu_0, x_1^2, x_1\mu_0, x_1^3, x_1^2\mu_0 \ldots \rangle$$

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Poincaré series equals

$$P(2, \infty) = q^2 + \frac{1 + q^8t^3}{1 - q^4t^2}.$$

There is some interesting 2-torsion

Recall that the Jones polynomial can be obtained from the HOMFLY-PT polynomial by the formula $J(q) = P(a = q^2, q)$.

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Conjecture (S. Gukov, N. Dunfield, J. Rasmussen)

Let $\mathcal{H}(K)$ denote the HOMFLY-PT homology of a knot K. Then there exists a differential d_2 on $\mathcal{H}(K)$ such that

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Theorem (J. Rasmussen)

There exists a spectral sequence from $\mathcal{H}(K)$ to $\mathrm{Kh}(K)$.

Conjecture (G., A. Oblomkov, J. Rasmussen, V. Shende)

The HOMFLY-PT homology of torus knot $T(m,n)$ can be modelled on finite-dimensional representations of **rational Cherednik algebra** with parameter $c = \frac{m}{n}$. The differential d_2 can be defined in terms of certain operators from this algebra.

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One can prove that at $m \to \infty$ this construction gives $\mathcal{H}_n = \lim_{m\to\infty} \mathcal{H}(T(m, n))$, and the differential d_2 in the limit coincides from the differential in the main conjecture.

L. Rozansky proved that $Kh(n, \infty)$ coincides with the homology of the categorified Jones-Wenzl projector, i.e. the S^n-colored $\mathfrak{sl}_2$ homology of the unknot.

These categorified projectors were studied by B. Cooper - V. Krushkal, I. Frenkel - C. Stroppel - J. Sussan.

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We can show that our complex $(\mathcal{H}_n, d_2)$ is quasi-isomorphic to the Cooper-Krushkal complex by constructing an explicit homotopy for $n = 1, 2, 3, 4$.

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Remark

For $n = 2$ one can also compare this construction to the following

Theorem (J. Przytycki)

$\mathrm{Kh}(2, \infty)$ *is the Hochschild homology of* $\mathrm{Kh}(1, \infty)$.

With $\mathbb{Z}_2$ coefficients we have $d_2(\xi_{2i}) = x_i^2$, and $d_2(\xi_{2i+1}) = 0$. Therefore

$$P(n, \infty) = \prod_{i=0}^{n-1} \frac{1 + q^{2i+4} t^{2i+1}}{1 - q^{2i+2} t^{2i}} \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1 - q^{4i+4} t^{4i}}{1 + q^{4i+4} t^{4i+1}}.$$

This agrees with the experimental data.

One can check that the homology of d_2 have nontrivial odd torsion. For example, we have the following result.

Theorem

Let $p > 3$ be a prime number. Then $H^*(\mathcal{H}_p, d_2)$ has $\mathbb{Z}_p$-torsion in bidegree $q^{2p+6}t^{2p}$.

The proof is explicit - we present an element m_p in $\mathcal{H}_n$ such that $d_2(m_p)$ is divisible by p.

It is useful to consider generating functions $x(z) = \sum_{i=0}^{n-1} x_i z^i$ and $\xi(z) = \sum_{i=0}^{n-1} \xi_i z^i$. Then d_2 can be rewritten as

$$d_2(\xi(z)) = x(z)^2.$$

Remark that $d_2(\dot{\xi}(z)) = 2x(z)\dot{x}(z)$.

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Consider the series

$$\mu(z) = \sum_{i=0}^{n-2} \mu_i z^i = x(z)\dot{\xi}(z) - 2\dot{x}(z)\xi(z).$$

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In ξ-degree 1 this follows from the recent theorem of B. Feigin who considered similar complexes in connection to the representation theory of the Virasoro algebra.

Conjecture

The Poincaré series for the stable Khovanov homology of (∞, ∞) torus knot is given by the formula

$$P(\infty,\infty) = \sum_{p=0}^{\infty} q^{2p^2} t^{2p(p+1)} (1 + q^{8p+12} t^{8p+5}) \times$$

$$\frac{(1+q^6t^3)(1+q^8t^5)\cdots(1+q^{2p+4}t^{2p+1})}{(1-q^2t^2)(1-q^4t^4)\cdots(1-q^{2p}t^{2p})}.$$

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The same formula without $(1+\ldots)$ factors describes the ξ-degree 0 part of the homology, and coincides with the LHS of extended Rogers-Ramanujan identity written by B. Feigin and A. Stoyanovsky. We have a conjectural formula for the finite n too.

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2. To describe the reduced Khovanov homology in this algebraic model, one should take the quotient by x_0 and ξ_0. The structure will be very similar to the unreduced case, and there is a relation between reduced (n, ∞) homology and unreduced $(n-2, \infty)$ homology.

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3. The $\mathfrak{sl}(m)$ stable homology is expected to be described by a similar construction: the differential d_2 is replaced by d_m given by the formula

$$d_m(\xi(t)) = x(t)^m.$$

Thank you.