Affine Springer fiber – sheaf correspondence

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Affine Springer fibers

Let G be a connected reductive group with Lie algebra $\mathfrak{g}$. Our main character is the **affine Springer fiber** associated to an element $\gamma \in \mathfrak{g}((t))$:

$$\mathrm{Sp}_\gamma = \left\{ g \in G((t)) \ : g^{-1}\gamma g \in G[[t]] \right\} / G[[t]]$$

which is a subvariety of affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$.

If γ is compact, regular and semisimple then Sp_γ is finite-dimensional, but usually singular and can have infinitely many irreducible components.

Affine Springer fibers in type A

In type A, the geometry of affine Springer fiber Sp_γ is controlled by the characteristic polynomial $f(t,\lambda) = \det(\lambda I - \gamma(t))$ and the **spectral curve** $C_\gamma = \{f(t,\lambda) = 0\} \subset \mathbb{C}^2_{t,\lambda}$.

Sp_γ is closely related to the **compactified Jacobian** of C_γ and the Hilbert scheme of points on C_γ.

The spectral curve C_γ defines an n-strand **braid** β which describe the behavior of eigenvalues of $\gamma(t)$ as t goes around the origin. The remarkable conjectures of Oblomkov, Rasmussen and Shende relate the homology of Sp_γ to triply graded **Khovanov-Rozansky homology** of β.

Examples

Example

The matrix $\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$ has characteristic polynomial $\lambda^2 - t^3$ and corresponds to a cuspidal curve in $\mathbb{C}^2$. The corresponding two-strand braid is $\beta = \sigma^3$ which closes to the trefoil knot. The affine Springer fiber Sp_γ is isomorphic to $\mathbb{CP}^1$.

Example

The matrix $\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ has characteristic polynomial $\lambda^2 - t^2$ and corresponds to a pair of lines in $\mathbb{C}^2$. The corresponding two-strand braid is $\beta = \sigma^2$ which closes to the Hopf link. The affine Springer fiber is the infinite chain of $\mathbb{CP}^1$.

Homology of Sp_γ

The centralizer of γ naturally acts on Sp_γ. If γ is quasihomogeneous, then Sp_γ admits as additional $\mathbb{C}^*$ action.

The equivariant Borel-Moore homology of Sp_γ was studied by Bezrukavnikov, Hikita, Lusztig, Varagnolo, Vasserot, Yun and others. In particular:

- $H_*(\mathrm{Sp}_\gamma)$ is a finitely generated module over $\mathbb{C}[T^*T^\vee]^W$ where $T^\vee$ is the Langlands dual torus. For $G = GL_n$, we get a module over $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm, y_1, \ldots, y_n]^{S_n}$.
- Hence, it defines a coherent sheaf on $T^*T^\vee/W$ which for $G = GL_n$ is simply $(\mathbb{C}^* \times \mathbb{C})^n/S_n$. It is supported on a certain Lagrangian subvariety.
- If γ is quasihomogeneous, then $H_*^{\mathbb{C}^*}(\mathrm{Sp}_\gamma)$ is a representation of the **spherical trigonometric Cherednik algebra** associated to G.

Infinite family

We will be interested in a family of affine Springer fibers

$$\mathrm{Sp}_\gamma,\ \mathrm{Sp}_{t\gamma},\ \mathrm{Sp}_{t^2\gamma}\dots$$

In type A, going from γ to $t\gamma$ corresponds to a **blowdown** of the spectral curve. For example, if the characteristic polynomial of γ is $f_\gamma = \lambda^n - t^m$ then the characteristic polynomial of $t^k\gamma$ is

$$f_{t^k\gamma} = \lambda^n - t^{m+kn}.$$

Topologically, going from γ to $t\gamma$ adds a **full twist** to the corresponding braid, so we are interested in a family of braids

$$\beta, \mathrm{FT}\,\beta, \mathrm{FT}^2\,\beta\dots$$

Main theorem

Theorem (G.,Kivinen,Oblomkov)

There is a graded algebra $\mathcal{A}_G = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$ (depending only on G) with the following properties:

- *For any γ the direct sum of homologies $M_\gamma = \bigoplus_k H_*(\mathrm{Sp}_{t^k\gamma})$ is a graded module over $\mathcal{A}_G$.*
- $\mathcal{A}_0 \simeq \mathbb{C}[T^* T^\vee]^W$.
- *$\mathcal{A}_1 \simeq \mathbb{C}[T^* T^\vee]^\epsilon$ where ϵ denotes the sign representation of W.*
- *For all $d \geq 0$, we have*

$$\mathcal{A}_d \simeq \mathbf{e}_d \bigcap_\alpha (1 - \alpha^\vee, y_\alpha)^d \subset \mathbb{C}[T^* T^\vee]$$

where α runs over the roots of $\mathfrak{g}$ and $\mathbf{e}_d$ is the projector to the ϵ^d-isotypic component.

Main theorem

As a corollary, γ defines a **quasicoherent sheaf** $\mathcal{F}_\gamma$ on the variety $X_G = \operatorname{Proj} \mathcal{A}_G$. Changing γ to $t\gamma$ corresponds to twisting $\mathcal{F}_\gamma$ with $\mathcal{O}(1)$.

The variety X_G is normal and admits a natural projection to $T^*T^\vee/W$, so one can think of it as a partial resolution of the commuting variety for $G^\vee$. However, for general G we expect that X_G is not smooth and $\mathcal{A}_G$ is not generated by $\mathcal{A}_0$ and $\mathcal{A}_1$. This is not a problem for $G = GL_n$:

Theorem (BFN; G.,Kivinen,Oblomkov)

For $G = GL_n$, the algebra $\mathcal{A}_G$ is generated in degrees 0 and 1, and $X_{GL_n} = \mathrm{Hilb}(\mathbb{C}^* \times \mathbb{C})$.

To sum up, to any $\gamma \in \mathfrak{gl}_n((t))$ we associate a sheaf $\mathcal{F}_\gamma$ on $\mathrm{Hilb}(\mathbb{C}^* \times \mathbb{C})$.

Examples: GL_n

Example

If γ is a diagonal scalar matrix with distinct eigenvalues, then $\mathcal{F}_\gamma$ is the **Procesi bundle** $\mathcal{P}$ on $\mathrm{Hilb}(\mathbb{C}^* \times \mathbb{C})$ restricted to a certain Lagrangian subvariety L.

Example

More generally, if $\gamma = \mathrm{diag}(s_1 t^k, \dots, s_n t^k)$ with s_i distinct and nonzero then $\mathcal{F}_\gamma = \mathcal{P} \otimes \mathcal{O}(k)$ restricted to L.

Example

Suppose that $\det(\lambda I - \gamma) = \lambda^n - t^{kn+1}$. Then $\mathcal{F}_\gamma$ is isomorphic to the restriction of $\mathcal{O}(k)$ to the punctual Hilbert scheme at $(1,0) \in \mathbb{C}^* \times \mathbb{C}$.

Examples: beyond GL_n

Let γ be an **equivalued** element of valuation $k \in \mathbb{Z}_{\geq 0}$, and G arbitrary. Then the results of Kivinen imply that

$$M_\gamma = J_\gamma / \mathbf{y} J_\gamma \quad \text{where } J_\gamma = \bigoplus_j \bigcap_\alpha (1 - \alpha^\vee, y_\alpha)^{k+j}$$

as a module over

$$\mathcal{A}_G = \bigoplus_j \mathbf{e}_d \bigcap_\alpha (1 - \alpha^\vee, y_\alpha)^d$$

The module structure is clear, and therefore M_γ yields a sheaf $\mathcal{F}_\gamma$ on $X_G = \operatorname{Proj} \mathcal{A}_G$ which generalizes $\mathcal{P} \otimes \mathcal{O}(k)$ to arbitrary type.

It would be very interesting to understand the variety X_G and this Procesi-like sheaf on it geometrically, and relate it to the work of Losev.

Main theorem

The above theorems admit a **quantization**:

Theorem (G.,Kivinen,Oblomkov)

There is a $\mathbb{Z}$-algebra $\mathcal{A}_G^\hbar = \bigoplus {}_i\mathcal{A}_j^\hbar$ with multiplication ${}_i\mathcal{A}_j^\hbar \otimes {}_j\mathcal{A}_k^\hbar \to {}_i\mathcal{A}_k^\hbar$ such that:

- *For any quasihomogeneous γ the direct sum $M_\gamma^\hbar = \bigoplus_k H_*^{\mathbb{C}^*}(\mathrm{Sp}_{t^k\gamma})$ is a graded module over $\mathcal{A}_G^\hbar$*
- *${}_i\mathcal{A}_i^\hbar$ is a spherical trigonometric Cherednik algebra for G with parameter depending on i*
- *${}_i\mathcal{A}_{i+1}^\hbar$ is the **shift** bimodule between ${}_i\mathcal{A}_i^\hbar$ and ${}_{i+1}\mathcal{A}_{i+1}^\hbar$*

At $\hbar = 0$, we get ${}_i\mathcal{A}_j^{\hbar=0} \simeq \mathcal{A}_{j-i}$ and the $\mathbb{Z}$-algebra degenerates into graded algebra.

Similar $\mathbb{Z}$-algebras built from shift bimodules for rational Cherednik algebras were studied by Gordon and Stafford. We define the shift bimodule in the trigonometric case (since studied by Wille Liu).

Action of antisymmetric polynomials

Since $\mathcal{A}_1$ is isomorphic to the space of W-antisymmetric polynomials, any antisymmetric polynomial defines an operator from $H_*(\mathrm{Sp}_\gamma)$ to $H_*(\mathrm{Sp}_{t\gamma})$. Such an operator can be constructed explicitly as follows:

- Consider the affine Springer fiber $\widetilde{\mathrm{Sp}_\gamma}$ in affine flag variety Fl_G
- The homology $H_*(\widetilde{\mathrm{Sp}_\gamma})$ has an action of $\mathbb{C}[T^*T^\vee] \rtimes W$
- One has $\left[H_*(\widetilde{\mathrm{Sp}_\gamma})\right]^W = H_*(\mathrm{Sp}_\gamma)$, $\left[H_*(\widetilde{\mathrm{Sp}_\gamma})\right]^\epsilon = H_*(\mathrm{Sp}_{t\gamma})[N]$
- An antisymmeric polynomial defines an operator $H_*(\widetilde{\mathrm{Sp}_\gamma})^W \to H_*(\widetilde{\mathrm{Sp}_\gamma})^\epsilon$.

However, constructing $\mathcal{A}_k$ for $k > 1$ and verifying relations for $\mathcal{A}_1 \cdot \mathcal{A}_1 \subset \mathcal{A}_2$ seems to be out of reach with this approach.

Coulomb branch algebras

Instead, we construct both $\mathcal{A}$ and $\mathcal{A}^\hbar$ as graded **Coulomb branch algebras** following Braverman, Finkelberg and Nakajima (BFN).

Consider the space

$$ {}_i\mathcal{R}_j = \left\{ [g,s] \in G((t)) \times^{G[[t]]} t^i \mathfrak{g}[[t]] \;:\; gs \in t^j \mathfrak{g}[[t]] \right\}, $$

then we define ${}_i\mathcal{A}^\hbar_j = H_*^{G[[t]] \rtimes \mathbb{C}^\times}({}_i\mathcal{R}_j)$. The multiplication ${}_i\mathcal{A}^\hbar_j \otimes {}_j\mathcal{A}^\hbar_k \to {}_i\mathcal{A}^\hbar_k$ is defined by the BFN convolution product.

Theorem (Braverman, Finkelberg, Nakajima; Webster)

The $\mathbb{Z}$-algebra $\mathcal{A}^\hbar$ is associative, and commutative for $\hbar = 0$. Furthermore, for all i and j the component ${}_i\mathcal{A}^\hbar_j$ is a free module over $H_*^{G\times\mathbb{C}^*}(\mathrm{pt})$.

Theorem (Kodera, Nakajima)

For all i the algebra ${}_i\mathcal{A}^\hbar_i$ is isomorphic to the spherical trigonometric Cherednik algebra with parameter depending on i.

BFN Springer theory

The algebra $\mathcal{A}$ acts on $M_\gamma = \bigoplus_k H_*(\mathrm{Sp}_{t^k\gamma})$ (or $\mathcal{A}^\hbar$ acts in $\mathbb{C}^*$-equivariant homology in quasihomogeneous case) by virtue of the **BFN Springer theory** developed by Hilburn-Kamnitzer-Weekes and Garner-Kivinen.

The action is defined using a certain convolution between ${}_i\mathcal{R}_j$ and $\mathrm{Sp}_{t^i\gamma}$, which is compatible with BFN convolution on ${}_i\mathcal{R}_j$. Thus, the multiplication in the algebra is compatible with the action on the module.

In other words, the action of $\mathcal{A}$ on M_γ is similar in spirit to the BFN Springer theory, but generalizes it to the $\mathbb{Z}$-algebra level. One still needs, though, **to identify the algebra $\mathcal{A}$ explicitly**.

Localization

The components ${}_i\mathcal{A}_j$ can be embedded into difference operators on $\mathbb{C}[\mathfrak{t}^{reg}]$ using localization techniques. Furthermore, the homology of ${}_i\mathcal{R}_j$ has a basis $[{}_i\mathcal{R}_j^{\leq\lambda}]$ corresponding to Schubert cells in the affine Grassmannian. Thus, we get an explicit basis of ${}_i\mathcal{A}_j$.

Theorem (G.,Kivinen, Oblomkov)

For arbitrary G the localization maps $[{}_i\mathcal{R}_j^{\leq\lambda}]$ to the difference operator

$$\sum_{\lambda'\in W\lambda} \frac{\prod_{\alpha(\lambda')+i<j}\prod_{\ell=0}^{i-\alpha(\lambda')-j-1}(y_\alpha+(\alpha(\lambda')+j+\ell)\hbar+c)}{\prod_{\alpha\in\Phi}\prod_{\ell=0}^{\max(0,\alpha(\lambda'))-1}(y_\alpha+\ell\hbar)} u_{\lambda'}+\dots$$

where $u^{\lambda'}$ is the translation by $\hbar\lambda'$ and $\dots$ are lower order terms with respect to some filtration.

Localization for GL_n

For $G = GL_n$ and $\hbar = 0$ the localization formula simplifies to

$$[{}_i\mathcal{R}_j^\lambda] \mapsto \pm\mathrm{Sym}\left(\Delta^{j-i} \prod_{r<s, |\lambda_r-\lambda_s|<|j-i|} (y_r - y_s)^{|j-i|-|\lambda_r-\lambda_s|} u^\lambda\right)$$

where Δ is the Vandermonde determinant in the y_i.

Example

Again for $G = GL_n$, assume that $j = i + 1$, then at $\hbar = 0$ we get

$$[{}_i\mathcal{R}_{i+1}^\lambda] \mapsto \pm\Delta\mathrm{Alt}\left(\prod_{r<s, \lambda_r=\lambda_s} (y_r - y_s) u^\lambda\right)$$

Up to $\pm\Delta$, this is antisymmetrization of a certain monomial in y_s and u_s, which agrees with the description of $\mathcal{A}_1$ as the space of antisymmetric polynomials.

Generation for GL_n

We have an easy combinatorial lemma:

Lemma

Suppose that λ is an arbitrary integral coweight for GL_n and $d > 0$. Then there exist d coweights $\mu^{(0)}, \ldots, \mu^{(d-1)}$ such that $\mu^{(0)} + \ldots + \mu^{(d-1)} = \lambda$ and for all i and j the following holds:

1) If $|\lambda_i - \lambda_j| < d$ then

$$d - |\lambda_i - \lambda_j| = \sum_{k, \mu_i^{(k)} = \mu_j^{(k)}} 1.$$

2) If $|\lambda_i - \lambda_j| > d$ then $\mu_i^{(k)} \neq \mu_j^{(k)}$ for all k.

By using the above formulas for the basis in ${}_i\mathcal{A}_j$, one can use this lemma to prove that the graded algebra $\mathcal{A}$ (resp. $\mathbb{Z}$-algebra $\mathcal{A}^\hbar$) is generated in degrees 0 and 1 for $G = GL_n$.

Further directions

Conjecture

The module $M_\gamma = \bigoplus_k H_*(\mathrm{Sp}_{t^k\gamma})$ *is finitely generated over* $\mathcal{A}_G$ *and the sheaf* $\mathcal{F}_\gamma$ *on* X_G *is coherent.*

The conjecture is known in several cases but open in general.

Problem

BFN define Coulomb branch algebras $\mathcal{A}_{G,N}$ *for arbitrary representations* N *of the group* G. *One can also define generalized affine Springer fiber for* $\gamma \in N((t))$. *What could one say about graded algebras and modules in this generality?*

Further directions

The work of G.-Neguț-Rasmussen, G.-Hogancamp and Oblomkov-Rozansky relates Khovanov-Rozansky homology to sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$. In particular, for any braids α, β there is a natural multiplication

$$\mathrm{HHH}(\alpha) \otimes \mathrm{HHH}(\beta) \to \mathrm{HHH}(\alpha\beta).$$

Given a braid β, one can form a graded module $\bigoplus_k \mathrm{HHH}(\beta\, \mathrm{FT}^k)$ over the graded algebra $\bigoplus_{k=0}^{\infty} \mathrm{HHH}(\mathrm{FT}^k)$, and a sheaf on $\mathrm{Proj} \bigoplus_{k=0}^{\infty} \mathrm{HHH}(\mathrm{FT}^k)$.

Theorem (G.,Hogancamp)

The variety $\mathrm{Proj} \bigoplus_{k=0}^{\infty} \mathrm{HHH}(\mathrm{FT}^k)$ *is isomorphic to the* **isospectral** *Hilbert scheme* $X_n(\mathbb{C}^2)$ *of* n *points on* $\mathbb{C}^2$.

Problem

Given $\gamma \in \mathfrak{gl}_n((t))$, *we can construct a sheaf* $\mathcal{F}_\gamma$ *on* $\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$ *using the methods in this talk, and another sheaf on* $X_n(\mathbb{C}^2)$ *using link homology. How are they related?*

Happy birthday, Professor Nakajima!