Equivariant Euler characteristics of the moduli spaces of pointed hyperelliptic curves

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The equivariant Euler characteristic equals

$$\chi^{S_n}(\mathcal{H}_{g,n}) = \sum_{i,\lambda} (-1)^i a_{i,\lambda} s_\lambda,$$

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$$\chi^{S_n}(\mathcal{H}_{g,n}) = \sum_i (-1)^i \sum_{\sigma \in S_n} (-1)^{|\sigma|} p_1^{k_1(\sigma)} \dots p_n^{k_n(\sigma)} \cdot \mathrm{Tr}\sigma|_{H^i(\mathcal{H}_{g,n})},$$

$k_i(\sigma)$ – number of cycles of length i in σ.

Specializations

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Also

$$\chi(\mathcal{H}_{g,n}/S_n) = \chi^{S_n}(\mathcal{H}_{g,n})| \quad p_i = 1,$$

$$\chi(\mathcal{H}_{g,n}/S_n, \pm 1) = \chi^{S_n}(\mathcal{H}_{g,n})| \quad p_i = (-1)^i.$$

- Genus 1, 2 – E. Getzler.

  Resolving mixed Hodge modules on configuration spaces. Duke Math. J. 96 (1999), no. 1, 175–203

  Euler characteristics of local systems on $\mathcal{M}_2$. Compos. Math. 132 (2002), 121–135

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  Euler characteristics of local systems on $\mathcal{M}_2$. Compos. Math. 132 (2002), 121–135

- Genus 3 – G. Bini, G. van den Geer.

  The Euler characteristic of local system on the moduli of genus 3 hyperelliptic curves. Math. Ann. 332 (2005), no. 2, 367–379

- Non-equivariant Euler characteristics – G. Bini

  The Euler characteristics of $\mathcal{H}_{g,n}$. Topology and its Applications, 155 (2007), 121–126.

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  The Euler characteristics of $\mathcal{H}_{g,n}$. Topology and its Applications, 155 (2007), 121–126.

- Point count over finite fields – J. Bergstrom, O. Tommassi

  O. Tommasi. Rational cohomology of the moduli space of genus 4 curves. Compos. Math. 141 (2005), no. 2, 359–384.

  J. Bergström, O. Tommasi. The rational cohomology of $\overline{\mathcal{M}}_4$. Math. Ann. 338 (2007), no. 1, 207–239.

  J. Bergström. Equivariant counts of points of the moduli space of the pointed hyperelliptic curves. arXiv:math.AG/0611813

Theorem

$$\sum_{k=0}^{\infty} t^k \chi^{S_k}(\mathcal{H}_{g,k}) = -\frac{1}{2\cdot 2g\cdot(2g+1)\cdot(2g+2)}[(1+p_1t)^{2-2g}+(1+p_1t)^{2+2g}(1+p_2t^2)^{-2g}]+$$

$$\sum_{n|(2g+1)}\frac{\varphi(n)}{2(2g+1)}[(1+p_1t)^3(1+p_nt^n)^{-\frac{2g+1}{n}}+(1+p_1t)^1(1+p_2t^2)(1+p_nt^n)^{\frac{2g+1}{n}}(1+p_{2n}t^{2n})^{-\frac{2g+1}{n}}]-$$

$$\sum_{n|(g+1),2|n}\frac{\varphi(n)}{4(2g+2)}[(1+p_1t)^4(1+p_nt^n)^{-\frac{2g+2}{n}}+(1+p_2t^2)^2(1+p_nt^n)^{-\frac{2g+2}{n}}]-$$

$$\sum_{n|(g+1),2\not| n}\frac{\varphi(n)}{4(2g+2)}[(1+p_1t)^4(1+p_nt^n)^{-\frac{2g+2}{n}}+(1+p_2t^2)^2(1+p_nt^n)^{\frac{2g+2}{n}}(1+p_{2n}t^{2n})^{-\frac{2g+2}{n}}]+$$

$$\sum_{n|2g+2,n\not| g+1}\frac{\varphi(n)}{2(2g+2)}(1+p_1t)^2(1+p_2t^2)(1+p_nt^n)^{-\frac{2g+2}{n}}-$$

$$\sum_{n|g,2\not| n}\frac{\varphi(n)}{4\cdot 2g}[(1+p_1t)^2(1+p_nt^n)^{-\frac{2g}{n}}+(1+p_1t)^2(1+p_nt^n)^{\frac{2g}{n}}(1+p_{2n}t^{2n})^{-\frac{2g}{n}}]-$$

$$\sum_{n|2g,2|n}(-1)^{1-\frac{2g}{n}}\frac{\varphi(n)}{2\cdot 2g}(1+p_1t)^2(1+p_nt^n)^{\frac{2g}{n}}(1+p_{2n}t^{2n})^{-\frac{2g}{n}}.$$

Everywhere we assume $n > 1$.

Theorem

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\chi(\mathcal{H}_{g,n})=\frac{-1}{2\cdot 2g(2g+1)(2g+2)}[(1+t)^{2-2g}+(1+t)^{2+2g}]$$

$$-\frac{g}{8(g+1)}[1+(1+t)^2]+\frac{g}{2g+1}[(1+t)+(1+t)^3]$$

$$+\frac{g+1}{4g}(1+t)^2.$$

Corollary

If $n > 2g+2$, then

$$\chi(\mathcal{H}_{g,n}) = (-1)^{n+1}\frac{(2g+n-3)!}{2\cdot 2g(2g+1)(2g+2)\cdot(2g-3)!}.$$

If $5 \leq n \leq 2g+2$, then

$$\chi(\mathcal{H}_{g,n}) = (-1)^{n+1}\frac{(2g+n-3)!}{2\cdot 2g(2g+1)(2g+2)\cdot(2g-3)!} - \frac{1}{2}\frac{(2g-1)!}{(2g+2-n)!}.$$

Also

$$\chi(\mathcal{H}_{g,0}) = 1, \chi(\mathcal{H}_{g,1}) = 2, \chi(\mathcal{H}_{g,2}) = 2, \chi(\mathcal{H}_{g,3}) = 0,$$
$$\chi(\mathcal{H}_{g,4}) = -2g, \chi(\mathcal{H}_{g,5}) = 0.$$

Structure of the answer:

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{H}_{g,n}) = \sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} \prod_j (1 + p_j t^j)^{k_j},$$

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$c_{k_1,\ldots,k_n}$ – some rational coefficients.

Key idea: $c_{k_1,\ldots,k_n}$ are orbifold Euler characteristics of certain spaces, and hence can be calculated.

Lemma

Suppose that a finite group G acts on a space X. Let $F(X,n)$ be the set of ordered n-tuples of distinct points of X. For $g \in G$ let $X_k(g)$ be the set of points of X with g-orbit of length k. Then

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(F(X,n)/G) = \frac{1}{|G|} \sum_{g \in G} \prod_{k} (1 + p_k t^k)^{\frac{\chi(X_k(g))}{k}}.$$

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Corollary

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\chi(F(X,n)/G) = \frac{1}{|G|}\sum_{g\in G}(1+t)^{\chi(X_1(g))}.$$

Consider the forgetful map

$$\pi_n : \mathcal{H}_{g,n} \to \mathcal{H}_g.$$

Its fiber is equal to

$$\pi_n^{-1}(C) = F(C,n)/Aut(C).$$

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Let Θ_G be the stratum in $\mathcal{H}_g$ of curves with $Aut(C) = G$. Then

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{H}_{g,n}) = \sum_G \chi(\Theta_g) \frac{1}{|G|} \sum_{g \in G} \prod_k (1 + p_k t^k)^{\frac{\chi(C_k(g))}{k}}.$$

We conclude that $c_{k_1,\ldots,k_n}$ is the orbifold Euler characteristic of the moduli space of pairs

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- $\chi(C_j(\varphi)) = jk_j$.

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Since C is hyperelliptic, structure of the orbits of φ can be reconstructed from the structure of the orbits of its restriction on $\mathbb{CP}^1\ldots$

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The orbifold Euler characteristics of this set equals

$$\frac{(-1)^{1-N/n}\varphi(n)}{N},$$

where $\varphi(n)$ *is the Euler function of* n*, i. e. the number of integers less than* n *and coprime with* n*.*

Theorem

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Everywhere we assume $n>1$.

The equivariant Euler characteristic of $\mathcal{M}_{g,n}$ (work in progress).

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Theorem. The generating function for the S_n-equivariant Euler characteristics of $\mathcal{M}_{g,n}$ has a form

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n}) = \sum_{m_j \geq 0 \text{ for } j<d, m_d<0} c_{m_1,\ldots,m_d} \prod_{j|d} (1+p_j t^j)^{m_j},$$

where the coefficients $c_{m_1,\ldots,m_d}$ are defined in the following way.

Let

$$c(d,j,\delta) = \mu(\frac{\delta}{(\delta,j)}) \frac{\varphi(d/j)}{\varphi(\delta/(\delta,j))}.$$

Define h by the equation (it should be integer)

$$\sum_{j\le d} m_j = 2-2h,$$

and let $s=\sum_{j<d} m_j$. Then

$$c_{m_1,\ldots,m_d} = \chi^{orb}(\mathcal{M}_{h,s})\cdot d^{2h-2}\cdot \frac{m_1!m_2!\ldots m_{d-1}!}{s!}\times$$

$$\sum_{m|d}\frac{\mu(m)}{m^{2h}}\sum_{\delta|d}\varphi(\delta)\prod_{j\,\vdots\, m,\ j<d} c(d,j,\delta)^{m_j},$$

where

$$\chi^{orb}(\mathcal{M}_{h,s}) = (-1)^s\frac{(2g-1)\cdot B_{2g}}{(2g-3)!}$$

is the orbifold Euler characteristic of $\mathcal{M}_{h,s}$.