Hilbert schemes of singular curves

Thursday, May 7, 2020 8:26 AM

C = curve $\subset \mathbb{C}^2$ $\{f(x,y)=0\}$

$\mathcal{O}_C = \dfrac{\mathbb{C}[x,y]}{(f)}$ ring of functions

$\mathcal{O}_{C,0} = \dfrac{\mathbb{C}[[x,y]]}{(f)}$ local ring at $(0,0)$

$\mathrm{Hilb}^n C = \{\text{ideals } I \subset \mathcal{O}_C : \dim \mathcal{O}_C/I = n\}$

$\cup$

$\mathrm{Hilb}^n(C,0) = \{\text{ideal } I \subset \mathcal{O}_{C,0} : \dim \mathcal{O}_{C,0}/I = n\}$

$\shortparallel$

ideals with support at 0

If C has one singularity at 0

we have stratification

$$\mathrm{Hilb}^n C = \bigsqcup_k \mathrm{Hilb}^k(C,0) \times \mathrm{Hilb}^{n-k}(C^{\text{smooth}})$$

<u>**Ex**</u> $C = \{y=0\}$ smooth $\mathcal{O}_C = \mathbb{C}[x]$

$\mathrm{Hilb}^n C = \{\text{ideals in } \mathbb{C}[x]\}$

$= \{(p(x))\}$ $\quad p(x) = (x-\alpha_1)\dots(x-\alpha_n)$

$\deg p = n$ $\quad = \underbrace{x^n + c_{n-1}x^{n-1} + \dots + c_0}$

c_i = coordinates on Hilb^n

$\mathrm{Hilb}^n \mathbb{C} = S^n \mathbb{C} = \mathbb{C}^n$

$\mathrm{Hilb}\,\mathbb{C} = S\,\mathbb{C} = \mathbb{C}$ on Hilb

Local: $\mathrm{Hilb}^n(C,0)$: ideals in $\mathbb{C}[[x]]$
$$\begin{array}{c} \| \\ (x^n) \end{array}$$

$\mathrm{Hilb}^n(C,0) = \{\text{point}\}$ for all n.

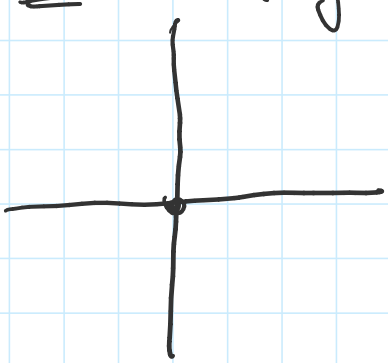
<u>Also</u> C is smooth but non-affine

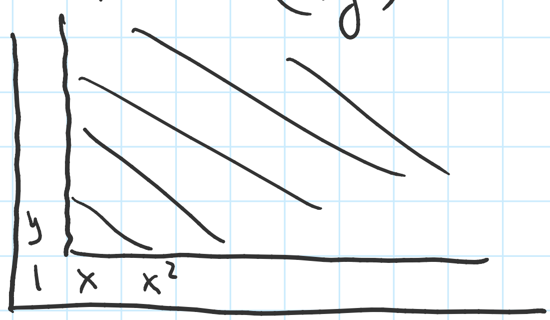
$\mathrm{Hilb}^n C = S^n C$ smooth, $\dim = n$

Macdonald's formula relates homology of C and of $\mathrm{Hilb}^n C$

<u>Ex</u> $C = \mathbb{CP}^1$ $\quad \mathrm{Hilb}^n(\mathbb{CP}^1) = S^n(\mathbb{CP}^1) = \mathbb{CP}^n$.

<u>Ex</u> $C = \{xy = 0\}$ $\qquad \mathcal{O}_{C,0} = \dfrac{\mathbb{C}[[x,y]]}{(xy)}$





$\mathrm{Hilb}^0(C,0) = pt = \{\mathcal{O}_{C,0}\}$

$\mathrm{Hilb}^1(C,0) = pt = \{\mathfrak{m}\}$ maximal ideal

$\mathrm{Hilb}^2(C,0) = \{(ax+by, x^2, y^2, xy)\}$
$$\begin{array}{c} \| \\ \mathbb{CP}^1 \end{array}$$

if a, b both $\neq 0$ this is

with $\mathbb{CP}^1$ coordinates $[a:b]$

if a,b both $\neq 0$ this is just $(ax+by)$

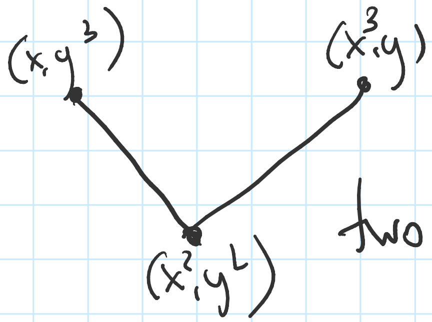
$a=0$ (y,x^2)

$b=0$ (y^2,x).

$$\mathrm{Hilb}^3(\mathbb{C},0) = \{ (ax+by^2), (ax^2+by)$$

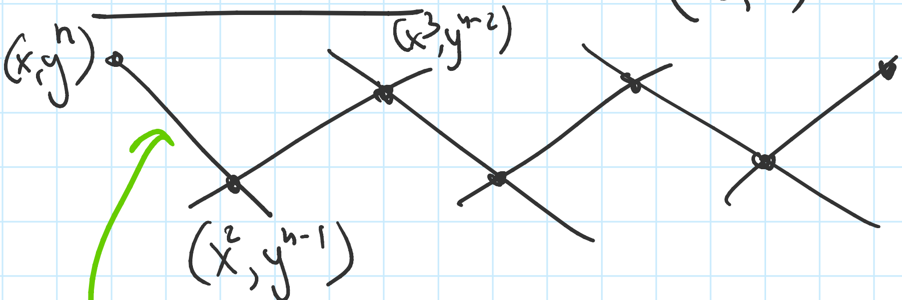
principal idals

and $(x,y^3), (x^2,y^2), (x^3,y) \}$



two $\mathbb{CP}^1$, glued at a point (~~it~~ intersecting transversally)

Can continue: $\mathrm{Hilb}^n(\mathbb{C},0)$ is a chain of $\mathbb{CP}^1$'s $(n-1)$



$(ax+by^{n-1})$
$(a,b\neq 0)$

Ex $C=\{xy=0\}\subset\mathbb{CP}^2$

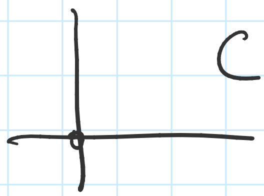
$\mathrm{Hilb}^n C$ has $n+1$ irreducible components isomorphic to $\mathrm{Bl}_{\mathbb{P}^{k-1}\times\mathbb{P}^{n-k-1}}(\mathbb{P}^k\times\mathbb{P}^{n-k})$

isomorphic to $Bl_{\mathbb{P}^{k-1}\times\mathbb{P}^{n-k-1}}(\mathbb{P} \times \mathbb{P} \quad)$

Can describe intersection of components ....

$n=2$ $\quad \mathbb{P}^2, \; Bl(\mathbb{P}^1\times\mathbb{P}^1), \; \mathbb{P}^2$ [Kivinen]

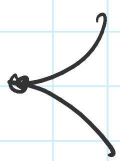
Can compute $H^*(\mathrm{Hilb}^n C)$ ....



C $\quad$ components:

$$\overline{\left\{ \begin{matrix} k \text{ points} \\ \text{on } \{x=0\} \end{matrix} \times \begin{matrix} n-k \text{ points} \\ \text{on } \{y=0\} \end{matrix} \right\}}$$

Ex $C = \{x^2 = y^3\} = \left\{ \begin{matrix} x = t^3 \\ y = t^2 \end{matrix} \right\}$



$\mathcal{O}_{C,0} = \mathbb{C}[[t^2, t^3]]$

$\mathrm{Hilb}^0(C,0) = pt$

$\mathrm{Hilb}^1(C,0) = pt$

$\mathrm{Hilb}^2(C,0) = \{ \underset{\mathbb{C}}{(t^2+\lambda t^3)}, (t^3,t^4) \} = \mathbb{C}P^1$

$\mathrm{Hilb}^3(C,0) = \{ (t^3+\lambda t^4), (t^4,t^5) \} = \mathbb{C}P^1$

So $\bullet, \circ, \mathbb{C}P^1, \mathbb{C}P^1, \ldots\ldots$ stabilize.

Fact If C is irreducible then $\mathrm{Hilb}^n(C,0)$ stabilize after $n \geq \mu$

$\mathrm{Hilb}^n(C,0)$ stabilize after $n \geq \mu$

and $\mathrm{Hilb}^n(C,0)$ is irreducible, for $n \geq \mu$

All this for plane curves.

$\dim \mathrm{Hilb}^n(C,0) = \delta = \frac{\mu}{2}$.

If C has several components, then $\mathrm{Hilb}^n(C,0)$ does not stabilize, number of comp. grows

$$\dim \mathrm{Hilb}^n(C,0) \leq \delta .$$

<u>Thm</u> (Altman, Kleiman, Iarrobino)

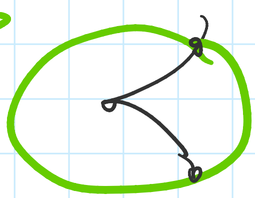
$\mathrm{Hilb}^n C$ is irreducible $\iff$ C irreducible <u>and</u> C has locally planar sing's.

<u>Problem:</u> Compute $H_*(\mathrm{Hilb}^n C)$ or $H_*(\mathrm{Hilb}^n(C,0))$

<u>Conj</u> [Oblomkov, Rasmussen, Shende]

$\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n(C,0))$ = Khovanov-Rozansky homology of the link of $(C,0)$

S^3



<u>Open problems</u>: (a) Is topology of $\mathrm{Hilb}^n(C,0)$ determined by topology of C

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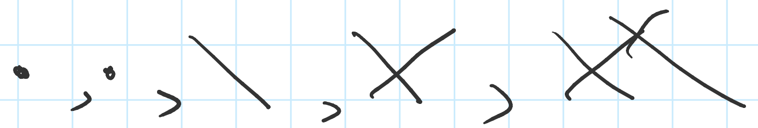
(i.e. just by Puiseaux pairs for 1 comp).

<u>Known</u>: for 1 Puiseaux pair there is an affine paving $\Rightarrow H_*(\mathrm{Hilb}^n(C,0))$

2 Puiseaux pairs, known for small examples but open in general.

(b) Is $\mathrm{Hilb}^n(C,0)$ paved by affine cells in general?

Generating function $\sum q^n t^k \dim H_k(\mathrm{Hilb}^n(C,0))$:

$\{xy=0\}$  , ...

$$1+q+q^2(1+t^2)+q^3(1+2t^2)+\dots = \frac{1}{1-q}+\frac{q^2t^2}{(1-q)^2}$$

$\{x^2=y^3\}$ $\bullet$, $\bullet$, $\mathbb{CP}^1$, $\mathbb{CP}^1$, ...

$$1+q+q^2(1+t^2)+q^3(1+t^2)+\dots = \frac{1+q^2t^2}{1-q}.$$

<u>Thm</u> C = projective curve with n components

(a) [Rennemo] If $h=0$, there is an action

(a) [Rennemo] If $n=0$, there is an action of $\langle x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ on $\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n C)$

and an action $\langle x, \frac{\partial}{\partial x} \rangle$ on $\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n(C,0))$

$$x : H_*(\mathrm{Hilb}^n) \longrightarrow H_*(\mathrm{Hilb}^{n+1})$$

$$y : H_*(\mathrm{Hilb}^n) \longrightarrow H_{*+2}(\mathrm{Hilb}^{n+1})$$

In particular, $\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n(C,0))$ is a module over $\mathbb{C}[x]$, always free.

This explains denominator $(1-q)$.

(b) [Kivinen] For n components, there is an action of $\langle \underbrace{x_1 \dots x_n, \frac{\partial}{\partial y_1} \dots \frac{\partial}{\partial y_n}, \sum \frac{\partial}{\partial x_i}, \sum y_i}_{2n+2 \text{ operators}} \rangle$

acting on $\bigoplus H_*(\mathrm{Hilb}^n C)$

and $\langle x_1, \dots, x_n, \sum \frac{\partial}{\partial x_i} \rangle$ on $\bigoplus H_*(\mathrm{Hilb}^n(C,0))$

In particular, $\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n(C,0))$ is a module over $\mathbb{C}[x_1 \dots x_n]$

For $\{xy=0\}$ $H_*(\mathrm{Hilb}^n(C,0)) = \frac{\mathbb{C}[x_1, x_2]}{(x_1-x_2)} \oplus \mathbb{C}[x_1, x_2]$

For $\{xy=0\}$ $H_\alpha(\mathrm{Hilb}(C,0)) = \underbrace{\frac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)}}_{= H_0} \oplus \underbrace{\mathbb{C}[x_1,x_2]}_{= H_2}$

<u>Note</u>: For global $\mathrm{Hilb}^n C$,

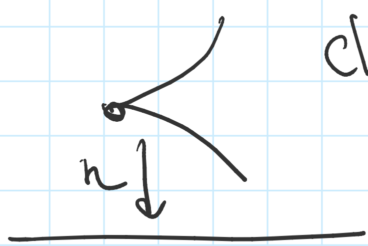
$\{xy=0\} \subset \mathbb{CP}^2$ the module structure

over $\langle x_1, x_2, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, y_1+y_2 \rangle$

is <u>interesting</u>:

$$\frac{\mathbb{C}[x_1,x_2,y_1,y_2]}{\mathbb{C}[x_1,x_2,y_1+y_2](x_1-x_2)}$$

<u>Parabolic Hilbert schemes</u>



Choose a projection to a line

Weierstrass preparation:

$f(x,y) = y^n + \varphi_{n-1}(x) y^{n-1} + \ldots + \varphi_0(x)$

$\Rightarrow \mathcal{O}_{C,0}$ has a basis

$\mathbb{C}[[x]]\langle 1, y, \ldots, y^{n-1}\rangle \simeq \mathbb{C}^n[[x]]$

n-dimensional space over $\mathbb{C}[[x]]$

y acts on this space as an operator:

$$Y = \begin{pmatrix} 0 & & & \varphi_0 \\ 1 & \ddots & 0 & \vdots \\ & \ddots & & \vdots \\ 0 & & 1 & \varphi_{n-1} \end{pmatrix}$$

$n \times n$ matrix

Char. polynomial of $Y = f(x,y)$.

matrix

Ideal in $\mathcal{O}_{C,0} \longleftrightarrow I \subset \mathbb{C}^n[[x]]$ subspace

s.t. $xI \subset I$

$YI \subset I$.

$v_1 \ldots v_n$ = basis of I over $\mathbb{C}[[x]]$

Y in this basis gets some form gYg^{-1}.

In particular $\dim I/xI = n$

$PHilb^{k,n+k}(C,0) = \{I_k \supset I_{k+1} \supset \ldots \supset I_{k+n} = xI_k\}$ ideals in $\mathcal{O}_{C,0}$

parabolic Hilbert scheme.

$PHilb^{k,n+k} \xrightarrow{\pi} Hilb^k$

$\{I_k \supset \ldots \supset xI_k\} \longrightarrow I_k$

[Better than considering all flags of ideals]

natural projection, fibers = complete flags in the space I_k/xI_k invariant under Y.

[Springer fiber]

fiber depend on Jordan structure of Y acting on I_k/xI_k.

$/x I_k$.

<u>Fact</u>: $H_*(\mathrm{PHilb}) \circlearrowleft S_n$

$$H_*(\mathrm{Hilb}) = H_*(\mathrm{PHilb})^{S_n}$$

We can study $H_*(\mathrm{PHilb})$ as a graded S_n-module.

$\mathcal{L}_i = I_{k+i-1} / I_{k+i}$ line bundles on PHilb

$$\{I_k \supset \dots \supset I_{k+n} = x I_k\} \xrightarrow{T} \{I_{k+1} \supset \dots \supset I_{k+n} = x I_k \supset I_{k+n+1} = x I_{k+1}\}$$

$$T : \mathrm{PHilb}^{k, n+k} \longrightarrow \mathrm{PHilb}^{k+1, n+k+1}$$

Now S_n, $c_1(\mathcal{L}_i)$, T_* act on $H_*(\mathrm{PHilb})$

<u>Thm</u> [G., Simental, Vazirani] $C = \{x^m = y^n\}$ $\gcd(m,n) = 1$

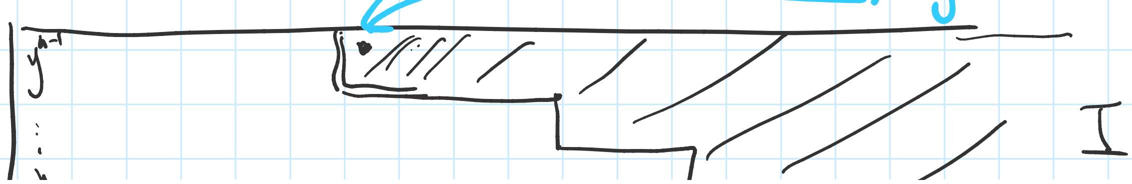
This gives an action of rational Cherednik algebra on $\oplus H_*^{\mathbb{C}^*}(\mathrm{PHilb})$. This is an irreducible representation.

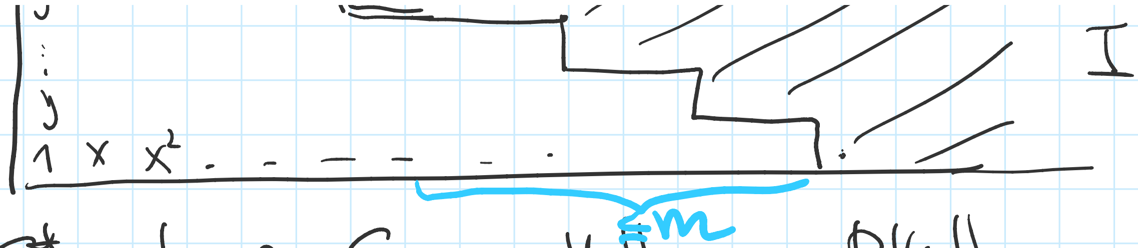
$\mathbb{C}^*$ equivariant homology.

<u>Combinatorics</u> in case $x^n = y^n$ $\circlearrowleft \mathbb{C}^*$

$$\mathcal{O}_{C,0} = \langle 1, \dots y^{n-1} \rangle \mathbb{C}[[x]]$$

$x^s y^{n-1}$





$\mathbb{C}^*$ acts on C, on Hilb, on PHilb, ...

if m, n coprime, fixed points = monomial ideals

I = all monomials to the right of staircase

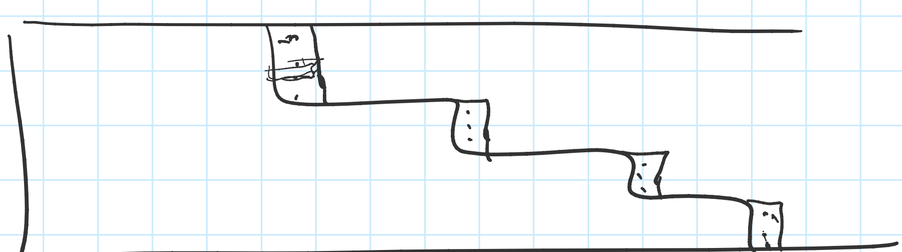
$$x^s y^{n-1} \cdot y = x^s y^n = x^s \cdot x^m = x^{s+m}$$

Therefore the width of the staircase $\leq m$

$$\left\{ \begin{matrix} \text{fixed pts} \\ \text{on Hilb} \end{matrix} \right\} = \left\{ \begin{matrix} \text{staircases in } n \times \infty \text{ strip} \\ \text{of width} \leq m \end{matrix} \right\}$$

Fixed points on PHilb

$I_k \supset I_{k+1} \supset \ldots \supset x I_k$ ← shift I_k by 1 to the right



flag ⟷ ordering of boxes between I_k and $x I_k$ s.t. invariant under y.

<u>Rmk</u> [G., Simental, Vazirani]

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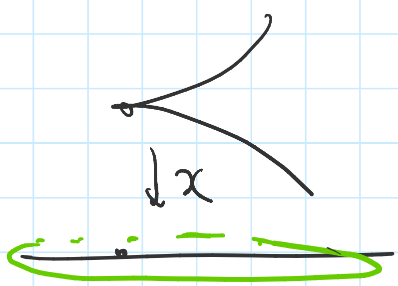
Explicit bijection between these fixed pts on PHilb and a basis in irred. representation of Cherednik algebra + find the action of all operators in this basis.

Q: How PHilb is related to topology?



Choice of projection $\longleftrightarrow$ "braid monodromy" of C

link of C = closure of an n-strand braid.

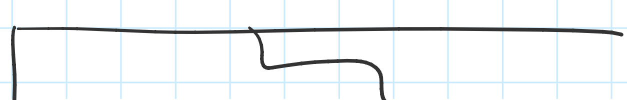
Conj [Trinh] $\bigoplus H_*(PHilb)$ (as graded S_n representation) is an explicit

or $\sum q^k \chi_{S_n}(PHilb^{k,k+n})$ topological invariant of the corresponding braid closure.

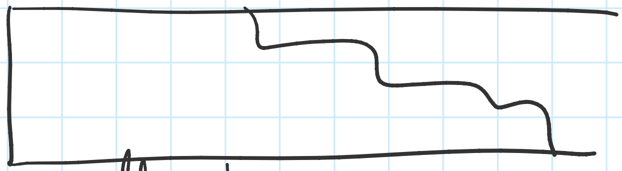
$\boxed{\text{symmetric function}}$

What if C is not reduced?

$\{y^n=0\}$



$\{y^n=0\}$



all staircases w. no restriction.

<u>Q</u>: $\{(x^2-y^3)^2=0\}$ multiplicity 2 cusp.

<u>Conj</u> $\{f^k=0\} \longleftrightarrow S^k$ colored homology of the link of f.

<u>Very recently</u>: [Hogancamp, Mellit '19]

(k) colored homology of $T(m,n)$ is supported in even homological degrees + explicit recursion.

Knot K + Young diagram $\lambda \rightsquigarrow P_\lambda(K)$.

<u>Question</u>: Is $\mathrm{Hilb}^\bullet(\{(x^m-y^n)^k=0\})$ paved by affine cells?