Cherednik algebras, Hilbert schemes and knot invariants

Eugene Gorsky
Stony Brook University

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Outline

I will present two different approaches to refined invariants of torus knots. They can be described:

I: As characters of certain representations of rational Cherednik algebras.
[E.G.-A.Oblomkov-J.Rasmussen-V.Shende; P.Etingof-E.G.-I.Losev]

II: As matrix elements of certain operators in the polynomial representation of double affine Hecke algebras.
[I. Cherednik; E.G.-A.Negut]

The approaches are conjecturally equivalent. Their equivalence can be proved in some special cases and gives rise to interesting combinatorial identities that can be verified on a computer.

Outline

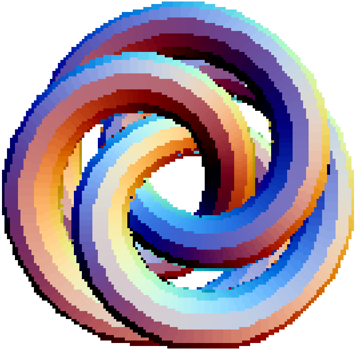
These two approaches match two approaches in mathematical physics:

I is expected to match the computations of the refined BPS invariants
[N. Dunfield-S. Gukov-J.Rasmussen; S.Gukov-M.Stosic]

II matches the refined Chern-Simons invariants
[M. Aganagic-S. Shakirov; A. Morozov et. al.]

Some connection to (colored) triply graded Khovanov-Rozansky homology is also expected.

(3,4) torus knot



Unrefined invariants

Unknot

Let us describe the unrefined invariants first. Given two coprime integers (m, n) and a Young diagram λ, one can construct *λ-colored HOMFLY invariant* $P^{\lambda}_{m,n}(a, q)$ of the (m, n) torus knot. At $a = q^N$ it specializes to the corresponding $sl(N)$ quantum invariant.

If $(m, n) = (1, 1)$, the knot is trivial and

$$P^{\lambda}_{1,1}(q^N, q) = P^{\lambda}(q^N, q) = s_{\lambda}(1, q, \ldots, q^{N-1}),$$

where s_{λ} is a Schur polynomial. This determines the rational function $P^{\lambda}(a, q)$ uniquely.

Unrefined invariants

Rosso-Jones formula

Theorem (M.Rosso-V.Jones)

One has

$$P^{\lambda}_{m,n}(a,q) = \sum_{\mu} q^{-\frac{m}{n}\kappa(\mu)} c^{\mu}_{\lambda,n} P^{\mu}(a,q),$$

where $\kappa(\mu) = \sum_{(i,j)\in\mu}(i-j)$ *is the content of* μ *and the coefficients* $c^{\mu}_{\lambda,n}$ *are defined by the equation*

$$s_{\lambda}(x_1^n, x_2^n, \ldots) = \sum_{\mu} c^{\mu}_{\lambda,n} s_{\mu}.$$

Rational Cherednik algebras

Definition

Let $\mathfrak{h}_n$ denote the Cartan subalgebra for $\mathfrak{sl}(n)$. Rational Cherednik algebra H_c of type $\mathfrak{sl}(n)$ is generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and the group algebra $\mathbb{C}[S_n]$ modulo following relations;

$$[x, x'] = 0, \quad [y, y'] = 0,$$

$$[x, y] = (x, y) - c \sum_{s \in \mathcal{S}} (\alpha_s^*, x)(\alpha_s, y)s,$$

where $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$, $\mathcal{S}$ is the set of all reflections in S_n and α_s is the equation of the fixed hyperplane for a reflection s.

Rational Cherednik algebras

Representations

Given a representation π_λ of S_n, one can define the action of H_c on

$$M_c(\lambda) := \pi_\lambda \otimes \mathbb{C}[\mathfrak{h}].$$

Irreducible representations $L_c(\lambda)$ are simple quotients of $M_c(\lambda)$.

All representations of H_c are naturally graded.

Theorem (Y. Berest-P. Etingof-V. Ginzburg)

The representation $L_c(\lambda)$ is finite-dimensional iff $\lambda = (n)$ and $c = m/n, GCD(m, n) = 1$.

Rational Cherednik algebras

Characters

Let m and n be coprime integers, λ be a Young diagram with d boxes. Define a space

$$\mathcal{H}^{\lambda}_{m,n} = \mathrm{Hom}_{S_{nd}}(\Lambda^{\bullet}\mathfrak{h}, L_{\frac{m}{n}}(n\lambda)).$$

It has q-grading induced from the grading on $L_{\frac{m}{n}}(n\lambda)$ and a-grading induced from the exterior degree on $\Lambda^{\bullet}\mathfrak{h}$.

Theorem

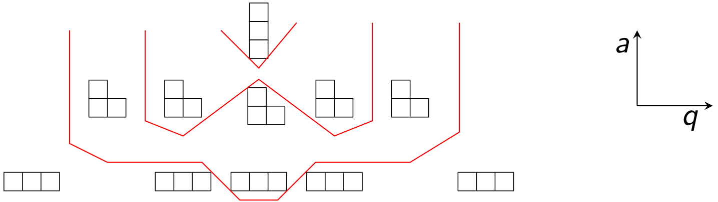
a)[E.G.-A.Oblomkov,J. Rasmussen-V.Shende] The bigraded character of $\mathcal{H}^{(1)}_{m,n}$ equals to $P^{(1)}_{m,n}$.

b)[P. Etingof-E.G.-I. Losev] The bigraded character of $\mathcal{H}^{\lambda}_{m,n}$ equals to $P^{\lambda}_{m,n}$.

Rational Cherednik algebras

Filtrations

Following the ideas of I. Gordon, in [GORS] we introduced a filtration on $L_{m/n}(n)$ with many useful properties. The character of the associated graded space to $\mathcal{H}_{m/n}$ is a 3-variable deformation of the HOMFLY polynomial. A filtration for general λ has yet to be constructed.



In this example we see a filtration on a 16-dimensional representation $L_{4/3}(3)$. The corresponding space $\mathcal{H}_{3,4}(3)$ has dimension 11.

DAHA

Main facts

The double affine Hecke algebra (DAHA) was introduced by I. Cherednik and played an important role in his proof of Macdonald's conjectures. We will use DAHA $\mathcal{A}_N$ of type GL_N. Its main features are:

- It contains two commutative subalgebras $\mathbb{C}[X_1, \ldots, X_N]$ and $\mathbb{C}[Y_1, \ldots, Y_N]$.
- It enjoys the action of $SL(2, \mathbb{Z})$ by algebra automorphisms
- It has a polynomial representation $V_N = \mathbb{C}[X_1, \ldots, X_N]$: X_i act by multiplication operators, and Y_i act by certain difference operators. In particular, $Y_1 + \ldots + Y_N$ is the Macdonald's operator D.

DAHA

Cherednik's polynomials

Following the physical constructions of M. Aganagic and S. Shakirov, I. Cherednik defined a remarkable family of polynomials by the following procedure

- Let M_λ be a Macdonald polynomial in X_i, considered as an element in $\mathcal{A}_N$
- Pick $K_{m,n} \in SL(2,\mathbb{Z})$ such that $K_{m,n}(1,0) = (m,n)$.
- Define $W^\lambda_{m,n} := K_{m,n}(M_\lambda) \in \mathcal{A}_N$
- Evaluate $W^\lambda_{m,n}$ in the polynomial representation:

$$\mathcal{P}^{\lambda,N}_{m,n}(q,t) = \varepsilon_N(W^\lambda_{m,n}(1)),$$

where $\varepsilon_N(f) = f(1,\ldots,q^{N-1})$.

Theorem (E.G.-A.Negut)

a)There is a polynomial $\mathcal{P}^{\lambda}_{m,n}(a,q,t)$ such that

$$\mathcal{P}^{\lambda,N}_{m,n}(q,t) = \mathcal{P}^{\lambda}_{m,n}(q^N,q,t).$$

b) $\mathcal{P}^{(1)}_{n,m}(a,q,t) = \frac{1}{(q-1)(t-1)} \times$

$$\times \sum_{\lambda=\square_1+\ldots+\square_n}^{SYT} \frac{\prod_{i=1}^{n} \chi_i^{S_{m/n}(i)} \cdot \frac{a-\chi_i}{1-\chi_i}}{\left(\frac{\chi_2 qt}{\chi_1} - 1\right) \ldots \left(\frac{\chi_n qt}{\chi_{n-1}} - 1\right)} \prod_{1\leq i<j\leq n} \omega\left(\frac{\chi_i}{\chi_j}\right)$$

where the sum is over all standard Young tableaux of size n, $\omega(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)}$, the constant χ_i denotes the q,t-content of the box $\square_i$ and: $S_{m/n}(i) = \left\lfloor \frac{im}{n} \right\rfloor - \left\lfloor \frac{(i-1)m}{n} \right\rfloor$.

DAHA

Idea of proof: Hilbert scheme

- O. Schiffmann and E. Vasserot proved that there is a well-defined $N \to \infty$ limit **SH** of the *symmetrized* algebras $\mathcal{A}_N$, which may be called "spherical DAHA for GL_∞". The Cherednik's operators and the polynomial representation can be appropriately renormalized and considered in this limit.
- The algebra **SH** is related to other interesting algebras: elliptic Hall algebra, shuffle algebra of Feigin-Odesskii.
- Schiffmann-Vasserot and Feigin-Tsymbalyuk constructed the action of **SH** on the equivariant K-theory of the Hilbert schemes of points on $\mathbb{C}^2$. A. Negut realized the operators $W^{\lambda}_{m,n}$ as certain geometric correspondences. The formula for $\mathcal{P}^{(1)}_{n,m}(a,q,t)$ comes from the equivariant localization of his construction.

DAHA

Example: (3,4) knot revisited

Up to some regrading, we have the following expression for $\mathcal{P}^{(1)}_{3,4}$ as sum over 4 standard tableaux of size 3:

$$\frac{t^6}{(t-q)(t^2-q)}(1+\frac{a}{t})(1+\frac{a}{t^2})-\frac{q^2t^2}{(q-t)(q^2-t)}(1+\frac{a}{t})(1+\frac{a}{q})$$

$$-\frac{q^2t^2}{(q-t)(q-t^2)}(1+\frac{a}{t})(1+\frac{a}{q})+\frac{q^6}{(q-t)(q^2-t)}(1+\frac{a}{q})(1+\frac{a}{q^2})$$

$$= q^3+qt+q^2t+qt^2+t^3+a\left(q+q^2+t+qt+t^2\right)+a^2.$$

The right hand side has 11 terms and matches the filtered character of $\mathcal{H}^{(1)}_{3,4}$.

Connection

Hilbert scheme

The relation between two approaches is expected to be of geometric nature: both are related to certain sheaves on the punctual Hilbert scheme of points on $\mathbb{C}^2$.

- A. Negut's geometric construction realizes the operator $W^{(1)}_{n,m}$ as a certain geometric correspondence on $\mathrm{Hilb}^i\,\mathbb{C}^2 \times \mathrm{Hilb}^{n+i}\,\mathbb{C}^2$. In particular, $W^{(1)}_{n,m}(1)$ is a vector in $K(\mathrm{Hilb}^n\,\mathbb{C}^2)$ realized by a certain explicit sheaf $\mathcal{F}_{m/n}$. The refined knot invariant can be computed as

$$\mathcal{P}(a,q,t) = \int_{\mathrm{Hilb}^n\,\mathbb{C}^2} \mathcal{F}_{m/n} \otimes \bigoplus_{i=0}^{n} a^i \Lambda^i T$$

  where T is the tautological rank n bundle on $\mathrm{Hilb}^n\,\mathbb{C}^2$.
- Results of I. Gordon and T. Stafford imply that to any (suitably filtered) representation of a rational Cherednik algebra one can associate a sheaf on $\mathrm{Hilb}^n\,\mathbb{C}^2$.

Connection

Example: $m = n + 1$

The case $m = n + 1$ was studied in details.

- The sheaf $\mathcal{F}_{(n+1)/n}$ coincides with the restriction of the line bundle $\Lambda^n T$ to the punctual Hilbert scheme
- It was studied by M. Haiman in connection with his work on diagonal harmonics. In particular, $\mathcal{P}^{(1)}_{n,n+1}(a = 0, q, t)$ coincides with the q, t-Catalan numbers of Garsia and Haiman
- On the other hand, I. Gordon constructed a filtration on $L_{(n+1)/n}$ and showed that it is related to $\mathcal{F}_{(n+1)/n}$ under the Gordon-Stafford correspondence.

Thank you.