Trace of the affine Hecke category
https://arxiv.org/abs/2201.07144

joint with Andrei Neguț (MIT)

Thm (Morton–Samuelson)

Skein algebra of T^2 $\simeq$ Elliptic Hall algebra at $t = q^{-1}$
(HOMFLY-PT) framed, oriented

Skein algebra: generators = links in $T^2 \times [0,1]$

relations = local skein relations

multiplication = stacking in $[0,1]$ direction.

Today: ① What is elliptic Hall algebra? Various presentations etc.

② How to categorify Morton–Samuelson Thm?

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① Elliptic Hall algebra (Burban–Schiffmann) $\mathcal{A} \circlearrowleft \widetilde{SL}(2,\mathbb{Z})$

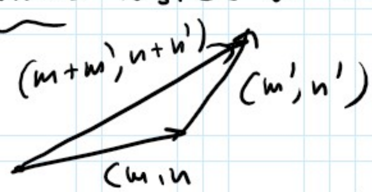
Generators: $P_{m,n}$ $(m,n) \in \mathbb{Z}^2$, we mostly focus on $n > 0$, m arbitrary

These correspond to torus links in Skein

Relations: complicated! In particular:

$$[P_{m,n}, P_{m',n'}] = (\text{coef})\, P_{m+m',n+n'}$$ depends on q, t.

if there are no integer points inside or on the boundary of the triangle



$\left(\iff \begin{vmatrix} m & m' \\ n & n' \end{vmatrix} = \pm 1\right)$

+ more relations if there are integer points on one side of triangle etc.

Facts: (a) At $t = q^{-1}$ the relations simplify dramatically:

$$[P_{m,n}, P_{m',n'}] = (q^d - q^{-d})\, P_{m+m',n+n'} \qquad d = \begin{vmatrix} m & m' \\ n & n' \end{vmatrix}$$

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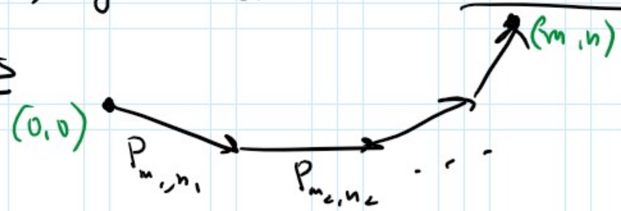
for <u>any</u> triangle! This was the starting point for Morton-Samuelson.

(b) The (positive part of) algebra $\mathcal{A}$ has a <u>PBW basis</u> given by <u>convex paths</u>



(c) The algebra $\mathcal{A}$ is bigraded, $\deg P_{m,n} = (m,n)$

dim (m,n) component $= \#$ convex paths from $(0,0)$ to (m,n)

$= \#$ compositions $(m,n) = (m_1,n_1) + (m_2,n_2) + \dots + (m_k,n_k)$ $\quad$ $,0$

$n_i > 0$, m_i any, $\sum m_i = m$, $\sum n_i = n$

up to permutation

<u>Shuffle Realization</u> (Feigin-Hashizume-Hoshino-Shiraishi-Yanagida, Feigin-Odesskii, Schiffmann-Vasserot, Negut)

$S_n \subset \mathbb{Q}(q,t)[z_1^{\pm}, \dots z_n^{\pm}]^{sym}$ consists of polynomials satisfying <u>wheel conditions</u> $f(z, qz, qtz, z_4 \dots z_n) = f(z, tz, qtz, z_4 \dots z_n) = 0$

$S = \bigoplus_{n \geq 0} S_n$ with shuffle product

$$F(z_1 \dots z_n) * G(z_1 \dots z_{n'}) =$$

$$= \mathrm{Sym}\left[\frac{F(z_1 \dots z_n) G(z_{n+1} \dots z_{n+n'})}{n!\, n'!} \prod_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq n+n'}} \frac{(1 - q\, z_i/z_j)(1 - t\, z_i/z_j)(1 - qt\, z_i/z_j)}{(1 - z_i/z_j)} \right]$$

$S_{n+n'}$

<u>Fact</u> Shuffle product preserves wheel condition and $S \simeq \mathcal{A}_{\geq 0}$ over $\mathbb{Q}(q,t)$.

<u>Why do we care?</u> It is easy to find more interesting

<u>Why do we care?</u> It is easy to find more interesting elements in $\mathcal{S}$, and study relations between them.

$\underline{d} = (d_1 \ldots d_n)$

$$R_{\underline{d}} = \mathrm{Sym}\left[\frac{z_1^{d_1} \cdots z_n^{d_n}(1-q)^{n-1}(1-t)^n}{\left(1-qt\frac{z_2}{z_1}\right)\cdots\left(1-qt\frac{z_n}{z_{n-1}}\right)} \prod_{i<j} \frac{\left(1-q\,{}^{z_i}/_{z_j}\right)\left(1-t\,{}^{z_i}/_{z_j}\right)\left(1-qt\,{}^{z_j}/_{z_i}\right)}{\left(1-{}^{z_i}/_{z_j}\right)}\right]$$

<u>Thm</u> (Neguț) (a) $R_{\underline{d}}$ satisfies wheel conditions, and hence defines an element in $\mathcal{S} \simeq \mathcal{A}$.

(b) For $d_i = \left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor$, $R_{\underline{d}} \longleftrightarrow P_{m,n}$

<u>Thm</u> (GN) The elements $R_{\underline{d}}$ satisfy the following relations:

$$R_{(d_1,\ldots d_i,d_{i+1},\ldots d_n)} - qt\,R_{(d_1,\ldots d_i-1,d_{i+1}+1,\ldots d_n)} = (1-q)R_{d_1\ldots d_i}R_{d_{i+1}\ldots d_n}$$

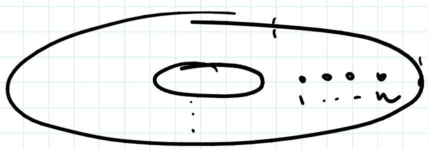
$$[\underset{\mathcal{S}_1}{R_K}, \underset{\mathcal{S}_n}{R_{(d_1,\ldots d_n)}}] = (t-1)\sum_{i=1}^{n}\begin{cases}\displaystyle\sum_{a=1}^{k-d_i} R_{(d_1\ldots d_{i-1},k-a,d_i+a,d_{i+1}\ldots d_n)} & \text{if } k \geq d_i \\ \displaystyle -\sum_{a=1}^{d_i-k} R_{(d_1\ldots d_{i-1},d_i-a,k+a,d_{i+1}\ldots d_n)} & \text{if } k \leq d_i\end{cases}$$

Furthermore, this is a complete set of relations over $\mathbb{Q}(q,t)$.

<u>Warning</u> If we work over $\mathbb{Z}[q^{\pm},t^{\pm}]$ instead of $\mathbb{Q}(q,t)$, we get different integral forms of $\mathcal{A}$.

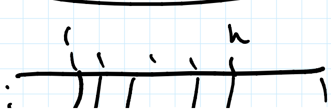
② <u>Affine braid group</u> <u>ABr_n</u>
(extended)

generated by $\underbrace{\sigma_1 \ldots \sigma_{n-1}}^{Br_n}$ and ω



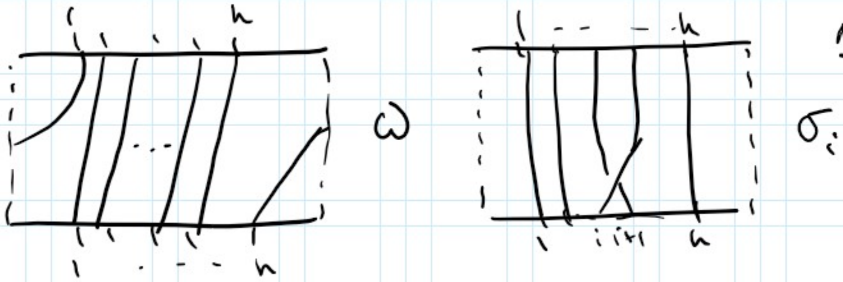
$\omega\,\sigma_i\,\omega^{-1} = \sigma_{i+1}$

<u>Note:</u>



$\omega\,\sigma\ \omega^{-1} = \sigma$

$\omega \sigma_i \omega^{-1} = \sigma_{i+1}$



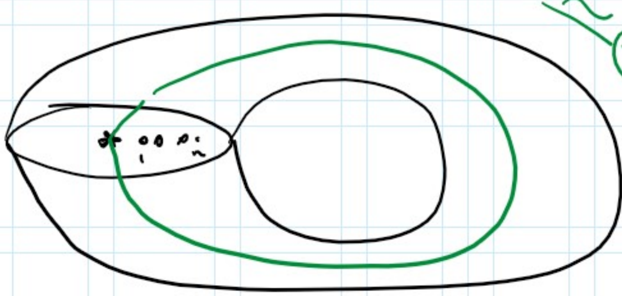
Note:
$\omega \sigma_{n-1} \omega^{-1} =: \sigma_0$
$\omega \sigma_0 \omega^{-1} = \sigma_1$

Define $y_i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1} \omega \sigma_{n-1} \cdots \sigma_i$, then $y_i y_j = y_j y_i$

$y_i \sim$ i-th travels around the annulus

Idea Links in $T^2 \times [0,1] \Longrightarrow$ Annular closures of affine braids

$\cong (D^2 - pt) \times S^1$



(~ conjugacy classes of affine braids)

$ABr_n \times ABr_m \longrightarrow ABr_{m+n}$ corresponds to skein multiplication for $T^2 \times [0,1]$

Categorification: $ABr_n \rightsquigarrow ASBim_n$ (extended) affine Soergel bimodules

Annular closure $\rightsquigarrow Tr(ASBim_n)$ horizontal trace.

$\tilde{R} = \mathbb{C}[x_1 \ldots x_n, \delta]$ Mackaay-Thiel, Elias

$B_i = \tilde{R} \otimes_{\tilde{R}^{s_i}} \tilde{R}$ $\quad T_i = [B_i \to \tilde{R}]$ Rouquier complexes

$\Omega_+ = \tilde{R}$, where left action standard

right action twisted by ω : $x_i \to x_{i+1}$
$x_n \to x_1 - \delta$

right action twisted by $\omega : x_i \longrightarrow x_{i+1}$

$x_n \longrightarrow x_1 - \delta$

<u>Thm</u> (Elias) (a) T_i and Ω satisfy the relations in ABr_n up to homotopy

(b) Commuting elements y_i lift to canonical complexes Y_i, commuting up to homotopy (Wakimoto objects) (Beliakova-Habiro-Lauda-Živković)

$Tr(ASBim_n)$ = horizontal trace of $ASBim_n$

$Tr : ASBim_n \longrightarrow Tr(ASBim_n)$ trace functor.

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③ <u>Results</u> Thm 1 (GN) The objects

(*) $Tr(Y_1^{d_1} \cdots Y_n^{d_n} T_w)$, w subword of $T_1 \cdots T_{n-1}$

generate $Tr(ASBim_n)$ as a triangulated category

<u>Rmk</u> $E_{\underline{d}} := Tr(Y_1^{d_1} \cdots Y_n^{d_n} T_1 \cdots T_{n-1})$

Then (*) is a skein product $E_{\underline{d}^{(1)}} * \cdots * E_{\underline{d}^{(r)}}$

Thm 2 (GN) $Tr(ASBim_n)$ is generated by $Tr(T_\sigma)$, T_σ = positive braid lift of minimal length representatives of conjugacy classes in $\widetilde{S}_n$, these are parametrized by <u>convex paths</u>

these are parametrized by convex paths

Ex $\mathrm{Tr}({}_{\llcorner}\Omega_1^m) \longleftrightarrow P_{m,n}$.

Thm 3 (GN) The objects $E_{\underline{d}}$ satisfy the following relations:

(a) There is a chain map

$$E_{d_1,\dots,d_i,d_{i+1},\dots,d_n} \longrightarrow E_{d_1,\dots,d_i-1,d_{i+1}+1,\dots,d_n}$$

with cone filtered by two copies of

$$E_{d_1,\dots,d_{i-1}} * E_{d_1,\dots,d_n}.$$

(b) For any $\underline{d}=(d_1,\dots,d_n)$ and k there is a sequence of objects $G_0,\dots,G_n$ in ASBim_n such that

- $\mathrm{Tr}(G_0)\simeq E_{(k)} * E_{\underline{d}}$, $\mathrm{Tr}(G_n)\simeq E_{\underline{d}} * E_k$
- There are maps $\begin{cases} G_{i-1}\longrightarrow G_i, & k\geq d_i \\ G_{i-1}\longleftarrow G_i, & k\leq d_i\end{cases}$

with cones filtered by $[q\mathbb{C}\xrightarrow{0}\mathbb{C}]\otimes$

$$\begin{cases} E_{d_1,\dots,d_{i-1},k-a,d_i+a,\dots,d_n} & k>d_i \\ E_{d_1,\dots,d_{i-1},d_i-a,k+a,\dots,d_n} & k<d_i\end{cases}$$

If $k=d_i$ then $G_{i-1}\simeq G_i$.

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(0) These relations match the relations between $R_{\underline{d}}$

Notes: (1) The exact sequences between $E_{\underline{d}}$'s

Notes: (1) The exact sequences between $E_{\underline{d}}$s categorify the skein relations in $\mathrm{Skein}(T^2)$ between the corresponding links

(2) One can use these to study $\mathrm{Tr}(AH_n) = \frac{AH_n}{[AH_n, AH_n]}$ where AH_n = affine Hecke algebra

(3) The exact sequences between $E_{\underline{d}}$'s match certain relations in $D_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Comm}_n)$ found by Neguț and Zhao (this was the main motivation for this work)

Here $\mathrm{Comm}_n = \{X, Y \in \mathrm{Mat}_{n\times n}, [X,Y]=0\}/GL_n$ is the <u>commuting stack</u>.

Alternatively, $E_{\underline{d}}$ are related to explicit correspondences between <u>Hilbert schemes of points on $\mathbb{C}^2$</u>

(4) Oblomkov-Rozansky theory provides yet another link between ABr_n and Comm_n/Hilb_n.

(5) <u>See the talk of Nicolle Gonzalez tomorrow!</u>