Cluster structures on braid varieties

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UCLA Combinatorics Seminar
October 6, 2022

Main result

For any braid β we define an affine algebraic variety $X(\beta)$ which we call **braid variety**.

Theorem (CGGLSS)

*$X(\beta)$ is a **cluster variety**. In other words, the algebra of functions $\mathbb{C}[X(\beta)]$ is a **cluster algebra**.*

- Parallel independent work by Galashin, Lam, Sherman-Bennett and Speyer.
- $X(\beta)$ can be defined for any simple Lie group G, but for most of the talk we will focus on $G = SL_n$.
- For a certain choice of β, $X(\beta)$ is isomorphic to an **open Richardson variety**. Above theorem resolves a conjecture of Leclerc on cluster structures on Richardson varieties.
- Leclerc's cluster structure was studied by Ménard, Keller–Cao and others. In special cases, conjecture has been proved by Serhiyenko–Sherman-Bennett–Williams and Galashin–Lam.

Braid varieties

The braid group Br_n has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| \geq 2).$$

Given a braid β, we can apply the braid relations and the move

$$\sigma_i \sigma_i \to \sigma_i$$

until we get a reduced expression for some permutation in S_n. The resulting permutation $\delta(\beta)$ does not depend on the choice of moves, it is called the **Demazure product** of β.

Example

We can simplify the braid $\beta = 1212$ in two ways:

$$1212 \to 1121 \to 121, \ 1212 \to 2122 \to 212 \to 121.$$

Braid varieties

We define matrices

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & z & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

They satisfy the identity

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_1z_3 - z_2)B_{i+1}(z_1).$$

Given a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ with Demazure product $\delta(\beta)$, we define the **braid variety**

$$X(\beta) = \{(z_1, \ldots, z_\ell) : \delta(\beta)^{-1} B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell) \text{ is upper} - \text{triangular}\}.$$

It is an affine algebraic variety in $\mathbb{C}^\ell$.

Braid varieties

A more geometric definition of $X(\beta)$ uses the flag variety parametrizing complete flags

$$\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots F_n = \mathbb{C}^n : \dim F_i = i\}.$$

Two flags $\mathcal{F}, \mathcal{F}'$ are in position σ_i, if $F_j = F_j'$ for $j \neq i$, and $F_i \neq F_i'$. We will denote this as $\mathcal{F} \xrightarrow{\sigma_i} \mathcal{F}'$.

Given a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ with Demazure product $\delta(\beta)$, we define the braid variety as the space of sequences of flags

$$X(\beta) = \{\mathcal{F}^0 \xrightarrow{\sigma_{i_1}} \mathcal{F}^1 \xrightarrow{\sigma_{i_2}} \mathcal{F}_2 \cdots \xrightarrow{\sigma_{i_\ell}} \mathcal{F}^\ell = \delta(\beta)\mathcal{F}^0\}$$

where $\mathcal{F}^0$ is a fixed standard flag.

Braid varieties

The braid variety has the following properties:

- Two definitions of $X(\beta)$ are equivalent.
- $X(\beta)$ is a smooth affine algebraic variety of dimension $\ell(\beta) - \ell(\delta(\beta))$.
- If β is reduced then $X(\beta)$ is a point.
- If β, β' are related by a braid move, $X(\beta) \simeq X(\beta')$.
- If $\beta = \cdots\sigma_i\sigma_i\cdots$ and $\beta' = \cdots\sigma_i\cdots$ then there is an open embedding

$$X(\beta') \times \mathbb{C}^* \hookrightarrow X(\beta).$$

- If $\beta(w), \beta(u^{-1}w_0)$ are positive braid lifts of permutations $w, u^{-1}w_0$ for $w > u$ then $X(\beta(w)\beta(u^{-1}w_0))$ is isomorphic to the **open Richardson variety** $R^{\circ}_{w,u}$.

Cluster varieties

A cluster variety is an affine algebraic variety X with the following structure:

- There is a collection of open charts $U \simeq (\mathbb{C}^*)^d$.
- Each chart U is equipped with **cluster coordinates** $A_1, \ldots, A_d$ which are invertible on U and extend to regular functions on X. These coordinates could be either **mutable** or **frozen**.
- To each chart one assigns a skew-symmetric integer matrix ε_{ij} or a **quiver** with $\max(0, \varepsilon_{ij})$ arrows from vertex i to vertex j.
- For each chart U and each mutable variable A_k, there is another chart U' with cluster coordinates $A_1, \ldots, A'_k, \ldots, A_d$ and a skew symmetric matrix ε'_{ij} related by **mutation** μ_k (see below).
- Any two charts in the collection are related by a sequence of mutations.
- The ring of functions on X is generated by all cluster variables in all charts.

The mutation μ_k is described by the equations

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{otherwise.} \end{cases}$$

and

$$A'_k = \frac{\prod_{\varepsilon_{ki} \geq 0} A_i^{\varepsilon_{ki}} + \prod_{\varepsilon_{ki} \leq 0} A_i^{-\varepsilon_{ki}}}{A_k}.$$

Mutation is involutive: $\mu_k^2 = \mathrm{id}$. It is clear that one chart with a specified set of cluster variables and the quiver determines all other charts.

As a warm-up example, consider the braid variety

$$X(\sigma^3) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_3 & -1 \\ 1 & 0 \end{pmatrix} \text{ upper-triangular} \right\} =$$

$$\{z_1 z_2 z_3 - z_1 - z_3 = 0\} = \{z_3(z_1 z_2 - 1) - z_1 = 0\} = \{z_1 z_2 - 1 \neq 0\}.$$

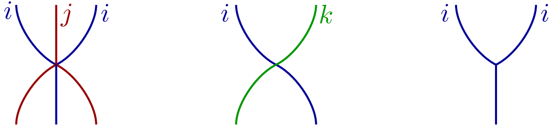
We have two charts:

- $\{z_1 \neq 0\}$ with coordinates $(A_1 = z_1, A_2 = z_1 z_2 - 1)$
- $\{z_2 \neq 0\}$ with coordinates $(A_1' = z_2, A_2 = z_1 z_2 - 1)$

Note that $A_1' = \frac{A_2+1}{A_1}$, so the two charts are related by mutation. The variable A_2 is frozen, while A_1 and A_1' are mutable.

Algebraic weaves

To describe some of the cluster charts, we use the formalism of algebraic weaves (or simply weaves). These are graphs with edges labeled by σ_i which are built from elementary pieces

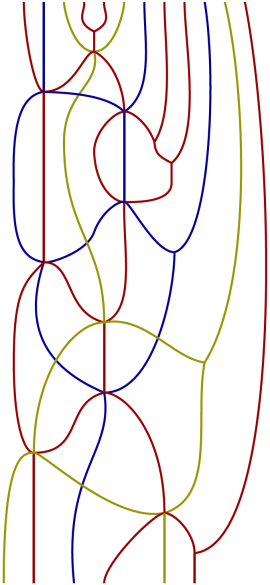


encoding braid moves and $\sigma_i \sigma_i \to \sigma_i$. Each horizontal section of a weave spells out a braid word, and we will always consider weaves with β on the top and $\delta(\beta)$ on the bottom.

Theorem

Each algebraic weave defines an open chart in $X(\beta)$ isomorphic to $(\mathbb{C}^*)^d$ where $d = \dim X(\beta)$ is the number of trivalent vertices.

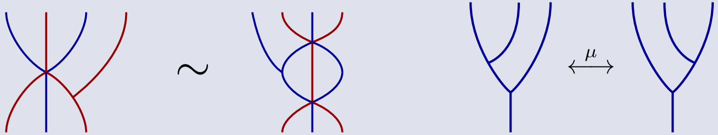
Algebraic weaves



Algebraic weaves

Theorem

a)[Elias] Any two weaves with the same braid words for β and $\delta(\beta)$ are related by a sequence of **weave equivalences** *and* **weave mutations***:*



b) Equivalent weaves define the same chart, but weave mutation changes a chart.

Main construction: Lusztig cycles

Let us fix a weave $\mathfrak{W}$. A cycle on $\mathfrak{W}$ is a function from the edges of $\mathfrak{W}$ to $\mathbb{Z}_{\geq 0}$. For a trivalent vertex v of $\mathfrak{W}$ we define a **Lusztig cycle** γ_v by the following rules:

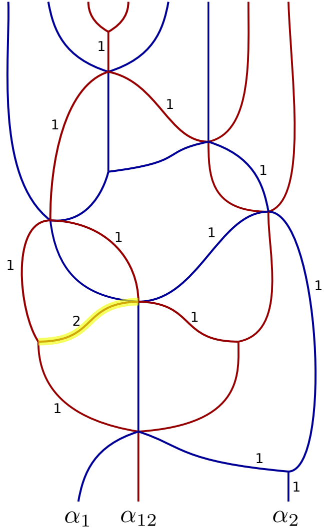
- All edges of $\mathfrak{W}$ which start above v have weight 0, and the outgoing edge of v has weight 1.
- For a trivalent vertex below v with incoming edges e_1, e_2 and outgoing edge e, we have $\gamma_v(e) = \min(\gamma_v(e_1), \gamma_v(e_2))$.
- For a 4-valent vertex with incoming edges e_1, e_2 and outgoing edges e_1', e_2', we have $(\gamma_v(e_1'), \gamma_v(e_2')) = (\gamma_v(e_2), \gamma_v(e_1))$.
- For a 6-valent vertex with incoming edges e_1, e_2, e_3 and outgoing edges e_1', e_2', e_3', we have

$$(\gamma_v(e_1'), \gamma_v(e_2'), \gamma_v(e_3')) = \Phi(\gamma_v(e_1), \gamma_v(e_2), \gamma_v(e_3))$$

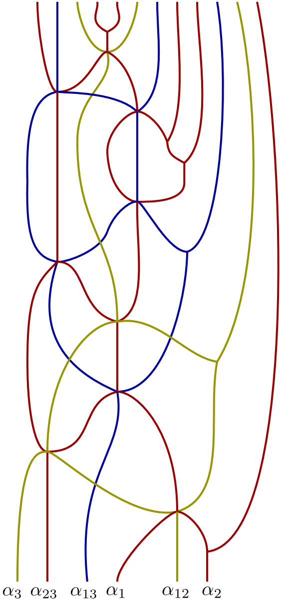
where $\Phi(a, b, c) = (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$.

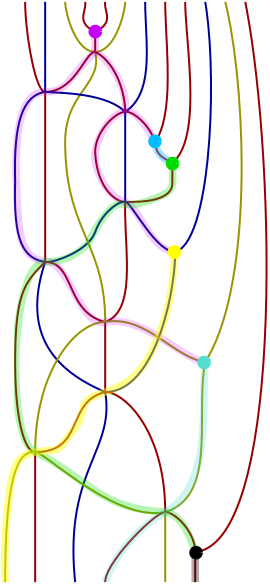
Main construction: Lusztig cycles

Here is an example of the Lusztig cycle γ_v for the topmost vertex v:



Main construction: Lusztig cycles





Main construction: quiver

For a given weave, the quiver and the matrix ε_{ij} is defined using the intersection form between cycles γ_v. The intersection number between two cycles C, C' is defined as the sum of local intersection numbers:

Definition (Local intersection at 3-valent vertex)

Suppose that C (resp. C') has weights a_1, a_2 (resp. b_1, b_2) on the incoming edges of a trivalent vertex v, and the weight a' (resp. b') on the outgoing edge. By definition, the local intersection number of C, C' at v is

$$\sharp_v(C \cdot C') = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a' & a_2 \\ b_1 & b' & b_2 \end{vmatrix}.$$

Main construction: quiver

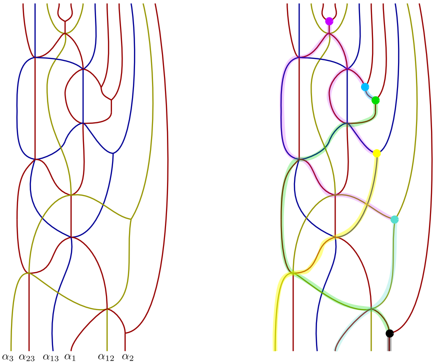
Definition (Local intersection at 6-valent vertex)

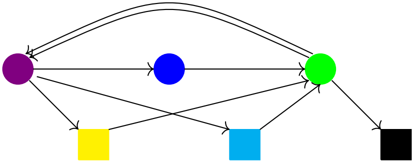
Suppose that C (resp. C') has weights a_1, a_2, a_3 (resp. b_1, b_2, b_3) on the incoming edges of a 6-valent vertex v, and weights a'_1, a'_2, a'_3 (resp. b'_1, b'_2, b'_3) on the outgoing edges. By definition, the local intersection number of C, C' at v is

$$\sharp_v (C \cdot C') = \frac{1}{2}\left(\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{vmatrix} \right).$$

Cycles which extend to the bottom correspond to frozen variables, other are mutable.

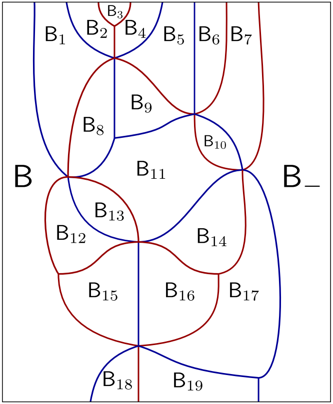
Main construction: quiver





Main construction: coordinates

We can label all regions of a weave by complete flags in $\mathbb{C}^n$. If two flags are separated by an edge of color i, they are in position σ_i. The flags on the top encode a point in $X(\beta)$.



Main construction: coordinates

Theorem

Given a weave $\mathfrak{W}$, there exists a unique collection of rational functions $A_v := A_v(z_1, \ldots, z_\ell) \in \mathbb{C}(z_1, \ldots, z_\ell)$, indexed by the trivalent vertices of the weave, and framed flags in the regions such that for every pair of regions r, r' separated by an edge e, the framed flags $B_r, B_{r'}$ are related by

$$B_i(\tilde{z})\chi_i\left(\prod_v A_v^{\gamma_v(e)}\right), \quad \text{for some } \tilde{z} \in \mathbb{C}.$$

Here $\chi_i(u) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & u & & & \\ & & & u^{-1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ and these A_v are **cluster variables**.

This completes the construction of cluster structure on $X(\beta)$.

Sketch of the proof

- We prove that weave equivalences do not change the quivers or cluster coordinates, while weave mutations correspond to cluster mutations.
- Since any two weaves for the same β are related by weave equivalences and mutations, all cluster charts for such weaves are mutation equivalent.
- For a braid word $\Delta\beta$ (where Δ is the half-twist), and *right inductive* weave, the cluster structure agrees with the one constructed by Shen-Weng for **double Bott-Samelson cells**.
- If $\beta' = \sigma_i\beta$, then the quiver for the *left inductive weave* for β is obtained from the one for β' by freezing some cluster variables.
- The variable z_1 is a cluster monomial for some weave. Rotation of a braid word corresponds to a quasi-cluster transformation, so all other z_i are cluster monomials. Therefore $\mathbb{C}[X(\beta)]$ is generated by cluster monomials.

Braid varieties in other types

For simply-laced types, the constructions and the proofs are very similar to type A, with minimal changes.

For non simply-laced types , there are more changes:

- Braid relations, weaves and weave equivalences look different. In particular, we will see $(2m_{ij})$-valent vertices in weaves.
- The rules for Lusztig cycles would look different. We use pairs of **Lusztig cycles** (for a group G) and **dual Lusztig cycles** (for the Langlands dual group $G^\vee$) similar to coroots and roots.
- The cluster exchange matrix is only skew-symmetrizable, but not skew symmetric.
- Still, many arguments can be deduced from the simply-laced case by folding.

Thank You!