# Parabolic Hilbert schemes on singular curves and representation theory

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$C$ = plane curve singularity
$\{f(x,y)=0\}$

$\mathcal{O}_C = \dfrac{\mathbb{C}[x,y]}{(f(x,y))}$ $\quad \mathcal{O}_{C,0} = \dfrac{\mathbb{C}[[x,y]]}{(f(x,y))}$

local ring at $(0,0)$

$\mathrm{Hilb}^n(C)$ = Hilbert scheme of $n$ points on $C$

$= \{\text{ideals } I \subset \mathcal{O}_C \mid \dim \mathcal{O}_C/I = n\}$

$\mathrm{Hilb}^n(C,0) =$ — // — ideals in $\mathcal{O}_{C,0}$

Goal for today: Understand the geometry of $\mathrm{Hilb}^n(C)$, $\mathrm{Hilb}^n(C,0)$ in particular, (equivariant) homology of these Hilbert schemes

Ex 1: $C$ = smooth curve

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$C=\{y=0\}$ $\quad \mathcal{O}_C = \mathbb{C}[x]$ PID

$\mathrm{Hilb}^n(C) = \{$principal ideals $(f)$ in $\mathbb{C}[x]$, $\dim \mathbb{C}[x]/(f) = n\}$

$= \{$monic polynomials of degree $n\} = \mathbb{C}^n$

In general, if $C$ is an arbitrary smooth curve, $\mathrm{Hilb}^n C = S^n C =$ smooth variety of dim. $n$

$C=\{y=0\}$ $\quad \mathcal{O}_{C,0} = \mathbb{C}[[x]]$

$\mathrm{Hilb}^n(C,0) = \{pt\} = \{(x^n)\}$

**Ex:** $C=\{x^2=y^3\}$ cusp

parametrize $x=t^3, y=t^2$

$\mathcal{O}_{C,0} = \mathbb{C}[[t^2,t^3]] \subset \mathbb{C}[[t]]$

$\mathrm{Hilb}^n(C,0)$:

| | | |
|---|---|---|
| $n=0$ | $\{\mathcal{O}_{C,0}\}$ | pt |
| $n=1$ | $\{m\}$ | pt |
| $n=2$ | $\{(t^2+\lambda t^3), (t^3,t^4)\}$ | $\mathbb{CP}^1$ |

$n=2 \quad \{(t^2+\lambda t^3),(t^3,t^4)\} \cup$

$\lambda \in \mathbb{C}$

$$n \geq 2 \quad \{(t^n+\lambda t^{n+1}),(t^{n+1},t^{n+2})\}$$

$\mathbb{C}P^1$

$$\mathrm{Hilb}^n(C,0) = \begin{cases} pt, & n \leq 1 \\ \mathbb{C}P^1, & n \geq 2 \end{cases}$$

If $C$ is more singular, then $\mathrm{Hilb}^n C$ and $\mathrm{Hilb}^n(C,0)$ are also very singular

<u>Fact</u> If $C$ is reduced, irreducible then $\mathrm{Hilb}^n(C,0)$ stabilize for $n \gg 0$

stabilizes to compactified Jacobian of $(C,0)$

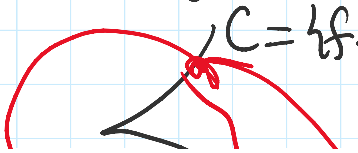
<u>Conj</u> (Oblomkov, Rasmussen, Shende)

$$\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n(C,0)) = (a=0) \text{ part}$$

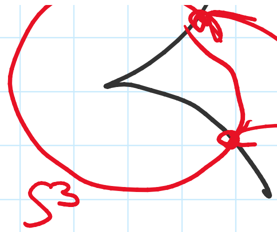
of the Khovanov–Rozansky homology of the link of $(C,0)$

$\mathbb{C}^2$

$C=\{f=0\}$



$S^3 = \{ \ldots$



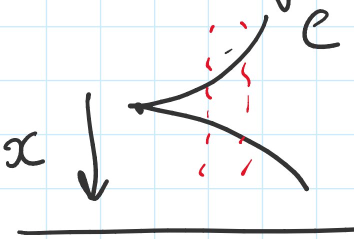

$S^3_\varepsilon \cap \{f=0\} = L$

link of $C$

**Ex** $C=\{x^2=y^3\} \rightsquigarrow L = $ trefoil knot

**Today:** Understand $\bigoplus_{n=0}^{\infty} H_*(\mathrm{Hilb}^n C)$ using geometric representation theory

Choose projection $x: C \longrightarrow \mathbb{C}$ of degree $k$




**Parabolic Hilbert scheme**

$$\mathrm{PHilb}^{k,n+k} = \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset I_{k+n} = xI_k\}$$

$I_s$ are ideals in $\mathcal{O}_C$, $\dim \mathcal{O}_C/I_s = s$

**Rmk** One can check $\dim I_k/xI_k = n$ for any ideal $I_k$.

More generally, we can consider

$$\mathrm{PHilb}^{\gamma,x} = \{\mathcal{O}_C \supset J^0 \supset J^1 \supset \dots \supset xJ^0\}$$

$J^s$ ideals, $\dim J^{s-1}/J^s = \gamma_s$

$J$-ideals, then $J/J^s = \gamma_s$

for some composition $(\gamma_1, \ldots, \gamma_r)$

"Compositional parabolic Hilbert scheme"

$$CPHilb^{n,x} = \bigsqcup_{\substack{\gamma \text{ with} \\ r \text{ parts}}} PHilb^{\gamma, x}$$

<u>Remark</u> $Hilb(C)$ is an invariant of $C$

$PHilb, \ldots$ <u>depend</u> on the choice of projection.

<u>Thm</u> (G., Simental, Vazirani) $C = \{x^m = y^n\}$
$GCD(m,n) = 1$.

Action of $\mathbb{C}^*$ on $C$: $(x,y) \to (\lambda^n x, \lambda^m y)$

$\rightsquigarrow$ action of $\mathbb{C}^*$ on all Hilbert schem, $PHilb, \ldots$

(a) $\bigoplus_k H_*^{\mathbb{C}^*}(PHilb^{k, n+k}(C))$ has action of rational Cherednik algebra with parameter $\frac{m}{n}$

(b) $\bigoplus_k H_*^{\mathbb{C}^*}(Hilb^k(C))$ has an action of <u>spherical</u> rational Cherednik algebra with param $\frac{m}{n}$

(c) In the limit $m \to \infty$, $\{y^n = 0\} = C_\infty$
<u>non-reduced</u> curve

Still, there an action of RCA/spherical RCA
with parameter "$c = \infty$"

Still, there an action of RCA/spherical RCA with parameter "$c=\infty$"

(d) $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y})$ has an action of quantized Gieseker algebra $\mathcal{A}_c(n,r)$ with parameter $c = \frac{m}{n}$.

compositional parabolic Hilb for projection $y$

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What are all these algebras?

(a) Rational Cherednik algebra (RCA) $H_c(n)$

generators $x_1 \ldots x_n, y_1 \ldots y_n, \mathbb{C}[S_n]$

commute commute

$i \neq j$ $[y_i, x_j] = c\,(i\,j)$ ← transposition in $\mathbb{C}[S_n]$

$[y_i, x_i] = 1 - c \sum_{j \neq i} (i\,j)$ [At "$c=\infty$" term 1 disappears]

(b) Spherical RCA: $eH_c(n)e$, where

$e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$

(c) $\mathcal{M}(n,r)$ = moduli space of rank $r$ torsion free sheaves on $\mathbb{C}^2$ with trivialization at $\mathbb{P}^1_\infty$ and $c_2 = n$

This is a smooth holomorphic symp. variety of dimension $2rn$.

of dimension $2rn$.

$\mathcal{A}_c(n,r)$ = quantization of $\mathcal{M}(n,r)$ =
= noncommutative deformation of the algebra of global functions $\mathbb{C}[\mathcal{M}(n,r)]$

Ex $r=1$ $\mathcal{A}_c(n,1) = eH_c(n)e$ spherical RCA

$n=1$ $\mathcal{A}_c(1,r)$ = certain quotient of $\mathcal{U}(\mathfrak{gl}(r))$

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Idea of proof: (a) $PHilb^{k,n+k} = \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset xI_k\}$

we need to construct an action of RCA

- Can define "Springer" action of $S_n$ using projection to $\{I_k \supset \dots \widehat{I_{k+s}} \supset \dots \supset xI_k\}$

- $\tau : PHilb^{k,n+k} \longrightarrow PHilb^{k+1,n+k+1}$

$\{I_k \supset I_{k+1} \supset \dots \supset xI_k\} \to \{I_{k+1} \supset \dots \supset xI_k \supset xI_{k+1}\}$

image of $\tau$ = {flags of ideals where divisible by $x$
= zero locus of some section of line bundle on $PHilb^{k+1,n+k+1}$

$\rightsquigarrow$ Gysin map: $H_*^{\mathbb{C}^*}(PHilb^{k+1,n+k+1}) \longrightarrow H_*^{\mathbb{C}^*}(PHilb^{k,n+k})$.

$\tau = x_1(1\dots n)$ $\qquad$ $\lambda = (1\dots n)^{-1}y_1$

$\ell = t_1(1 \dots n) \qquad x = (1 \dots n)\, y_1$

Need to check the relation ....

(d) Uses a recent result of Etingof-Krylov-Losev

$$L_{\frac{m}{n}}(n,r) = \left(L_{\frac{n}{m}} \otimes (\mathbb{C}^{r})^{m}\right)^{S_m} \text{-} \text{isotypical}$$

irrep of $\mathcal{A}_c(n,r)$

irrep of $H_{\frac{n}{m}}(m)$ RCA

swap $m \leftrightarrow n$

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## Coulomb branches

$G$ = reductive algebraic group

$N$ = representation of $G$

Braverman-Finkelberg-Nakajima defined a Coulomb branch algebra $\mathcal{A}(G,N)$ defined as equivariant Borel-Moore homology of a certain space related to affine Grassmannian of $G$.

- $G = GL(n)$, $N = \mathfrak{gl}(n) \oplus \mathbb{C}^n$

Kodera-Nakajima: $\mathcal{A}(G,N)$ = spherical RCA $eH_c(n)e$

- $G = GL(n)^{\times r}$, $N = \mathfrak{gl}(n)^{\oplus r} \oplus \mathbb{C}^n$

Nakajima-Takayama: $\mathcal{A}(G,N) = \mathcal{A}_c(n,r)$

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quantized Gieseker algebra.

<u>Thm</u> (Hilburn-Kamnitzer-Weekes) For arbitrary $G, N$, the BFN algebra $\mathcal{A}(G,N)$ acts on the homology of <u>any</u> generalized affine Springer fiber $X_v$ for $(G,N)$, specified by a choice of a vector $v \in N((t))$ (satisfying some mild assumptions).

<u>Thm</u> (Garner-Kivinen) For <u>any</u> plane curve singularity there is a choice of $v \in \mathfrak{gl}(n) \oplus \mathbb{C}^n$ such that $X_v = \bigsqcup_k \mathrm{Hilb}^k(C)$
generalized affine Springer fiber for $(GL(n), \mathfrak{gl}(n) \oplus \mathbb{C}^n)$

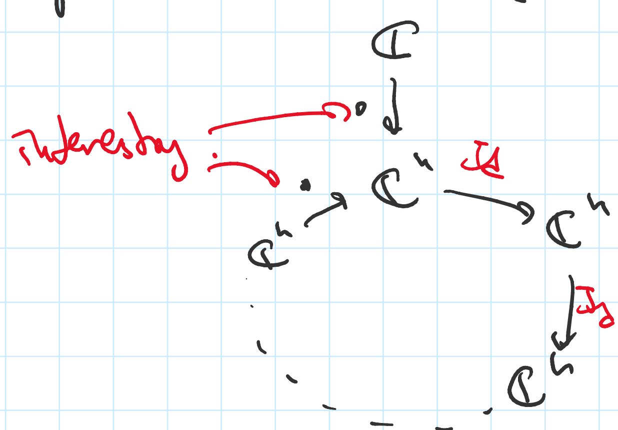
<u>Rmk</u> The equation of the curve = characteristic polynomial of the $\mathfrak{gl}(n)$-component of $v$.

<u>Cor</u> (a) If $C$ has a $\mathbb{C}^*$ action (like $x^m = y^n$) then $\bigoplus_k H_*^{\mathbb{C}^*}(\mathrm{Hilb}^k(C))$ has an action of spherical RCA

(b) If $C$ is general action of $\mathbb{C}[x_1 \dots x_n, y_1 \dots y_n]^{S_n}$

(b) If $C$ is general, action of $\mathbb{C}[x_1 \dots x_n, y_1 \dots y_n]^{S_n}$ in $\bigoplus_k H_*(\mathrm{Hilb}^k(C))$

<u>Thm</u> (G., Simental, Vazirani) For any $C$, $CP\mathrm{Hilb}^{r,y}(C,0)$ is the generalized affine Springer fiber for $(G = GL(n)^{\times r}, N = \mathfrak{gl}(n)^{\oplus r} \oplus \mathbb{C}^n)$




$N$ = space of all morphisms here

$r$ copies of $\mathbb{C}^n$

$v$ = choose $r-1$ arrows to be Id.

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<u>Related</u>: Oblomkov–Yun: trigonometric Cherednik algebra acts on homology of affine Springer fiber / compactified Jacobian.

Can get action of RCA using "perverse" filtration.