Symmetric functions and link invariants

Eugene Gorsky
University of California, Davis

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Symmetric functions are everywhere in modern mathematics.

In this talk, I will focus on interactions between symmetric functions and low-dimensional topology. Theory of symmetric functions helps to compute and understand various invariants of knots and links. Conversely, ideas from topology and categorification clarify many identities and properties of symmetric functions.

In other words, I will **draw** a lot of various symmetric functions.

Symmetric functions

A function in N variables $x_1, \ldots, x_N$ is called **symmetric** if it does not change under permutations of variables. For example (for $N = 3$):

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$

$$h_3 = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3$$

$$p_4 = x_1^4 + x_2^4 + x_3^4$$

In general, the **elementary symmetric function** e_k is the sum of all products of k distinct variables, the **complete symmetric function** h_k is the sum of all homogeneous monomials of degree k and the **power sum** p_k is the sum of x_i^k.

Symmetric functions

Theorem (Newton)

The ring of symmetric functions is freely generated by $e_1, \ldots, e_N$, or by $h_1, \ldots, h_N$, or by $p_1, \ldots, p_N$.

Schur functions are defined as

$$s_\lambda = \frac{\det\left(x_j^{\lambda_i+N-i}\right)}{\prod_{i<j}(x_i - x_j)}$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N)$ is a partition.

We can also consider symmetric functions in infinitely many variables.

Symmetric functions

We will need two operations of **plethysm**: given a symmetric function f, we define

$$f\left[\frac{X}{1-q}\right] \text{ by substituting } p_k \mapsto \frac{p_k}{1-q^k}$$

and

$$f[X(1-q)] \text{ by substituting } p_k \mapsto p_k(1-q^k)$$

For example,

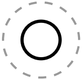
$$h_2 = \frac{p_1^2 + p_2}{2},$$

so

$$h_2[X(1-q)] = (1-q)^2\frac{p_1^2}{2} + (1-q^2)\frac{p_2}{2} = (1-q)(h_2 - qe_2).$$

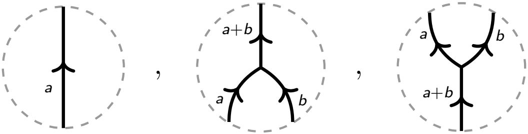
The HOMFLY-PT invariant of links is defined by the following rules:

 $-$  $= (q - q^{-1})$ .

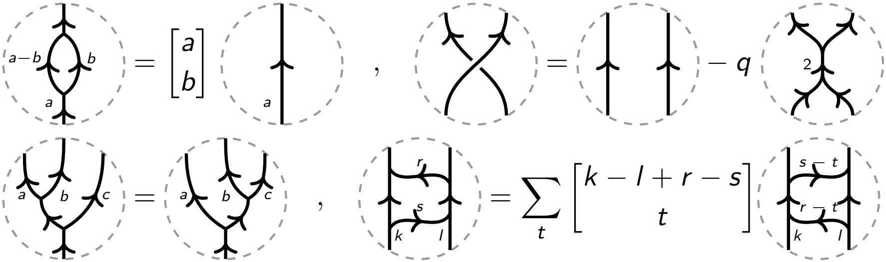
 $= \dfrac{a - a^{-1}}{q - q^{-1}}$  ,  $= -a^{-1}$ 

Given an (oriented) link diagram in the plane, we can use these rules to simplify it until the link becomes trivial.

It is possible to extend skein relations to **webs**, planar labeled trivalent graphs which are built of local pieces:

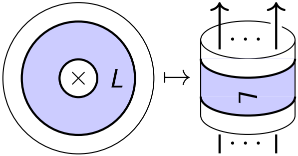


modulo generalized skein relations, in particular,



We will be interested in **annular links**, that is, links in the thickened annulus (or in the punctured plane). For example, a braid closure in the annulus is an annular link.

Annular links can be used to describe various "cabling" or "wrapping" operations:

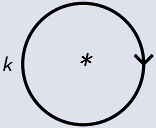


Symmetric functions from skein theory

Theorem (Turaev, Morton,..)

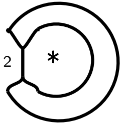
Link diagrams[a] *in the annulus (or in the punctured plane) modulo skein relations form an infinite-dimensional space, isomorphic to the space of symmetric functions in infinitely many variables. Under this isomorphism, elementary symmetric functions correspond to unknotted essential circles:*

$e_k =$



[a] We assume that all strands are oriented clockwise

What about more complicated webs/links in the annulus?



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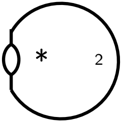
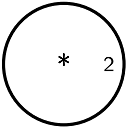
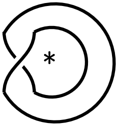


$= (q+q^{-1})$



$= (q+q^{-1})e_2$



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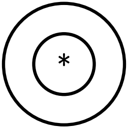
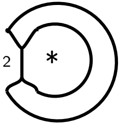


$-\,q$



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$$e_1^2 - q(q+q^{-1})e_2 = h_2 - q^2 e_2 = \frac{1}{1-q^2} h_2[X(1-q^2)]$$

Symmetric functions from skein theory

Theorem (G.,Wedrich)

Any web in the annulus corresponds to a **Schur positive** *symmetric function.*

As we saw, symmetric functions corresponding to knots and links are usually not Schur positive. However, they often become Schur positive after applying a plethysm:

Theorem (G.,Wedrich)

Let F_ϵ be the symmetric function corresponding to the braid $\sigma_1^{\epsilon_1}\cdots\sigma_{n-1}^{\epsilon_{n-1}}$, $\epsilon_i=\pm 1$. Then $(1-q^2)F_\epsilon\left[\frac{X}{1-q^2}\right]$ is the ribbon skew Schur function, *in particular, it is Schur positive.*

Theorem (G.,Oblomkov, Rasmussen, Shende)

Let $F_{m,n}$ be the symmetric function corresponding to the (m,n) torus knot. Then $(1-q^2)F_{m,n}[\frac{X}{1-q^2}]$ is Schur positive.

Categorification

These positivity results follow from the **categorification** of the skein of the annulus. It is closely related to annular Khovanov homology (Asaeda-Przytycki-Sikora, Beliakova-Putyra-Wehrli, Licata-Grigsby-Wehrli), annular Khovanov-Rozansky (Queffelec-Rose, Robert-Wagner) and derived categorical traces (G.-Hogancamp-Wedrich).

In short, there is a category where objects correspond to annular webs and links, and morphisms correspond to certain singular surfaces (foams) between them. We prove the following:

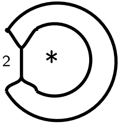
Theorem (G., Hogancamp, Wedrich)

Consider the object E_n corresponding to n concentric circles. Then:

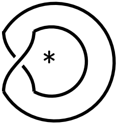
- *The endomorphism algebra of E_n is isomorphic to $\mathbb{C}[x_1, \dots, x_n, \theta_1, \dots, \theta_n] \rtimes S_n$ where x_i are even and θ_i are odd variables.*
- *The direct summands of E_n categorify Schur functions and generate the annular category.*

Categorification

For example, for $n = 2$ we have $\mathrm{End}(E_2) = \mathbb{C}[x_1, x_2, \theta_1, \theta_2] \rtimes S_2$. The action of S_2 permutes two circles, and we can consider symmetric and antisymmetric summands S^2E, $\Lambda^2 E$.



$$= (q + q^{-1})\Lambda^2 E$$



$$= \left[q^2\Lambda^2 E \xrightarrow{x_1 - x_2} S^2E \right].$$

Here powers of q indicates grading shifts.

Symmetric functions from quantum groups

Reshetikhin and Turaev defined link invariants using representation theory of quantum groups. We will focus on the quantum group $U_q(\mathfrak{gl}_N)$. Here are the key facts from Reshetikhin-Turaev theory:

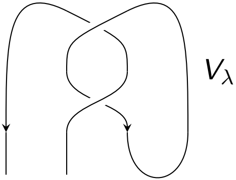
- Given a knot K and a representation V of $U_q(\mathfrak{gl}_N)$, one can define a topological invariant $\mathrm{RT}(K; V)$ (function of q)
- For the 2-dimensional representation of $U_q(\mathfrak{gl}_2)$, this invariant agrees with the Jones polynomial
- More generally, for the N-dimensional representation of $U_q(\mathfrak{gl}_N)$, this invariant agrees with the specialization of HOMFLY-PT at $a = q^N$
- If a knot K is cut open at one place, one can define the **universal invariant** $J_k \in \mathcal{Z}(U_q(\mathfrak{gl}_N))$ such that

$$\mathrm{Tr}(J_K|_V) = \mathrm{RT}(K; V).$$

Reshetikhin-Turaev invariants are related to the skein theory via **quantum Schur-Weyl duality**.

Symmetric functions from quantum groups

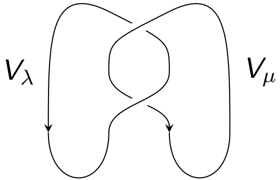
By taking characters of representations of $U_q(\mathfrak{gl}_N)$, one can identify these with symmetric functions in N variables. In particular, Schur polynomials s_λ correspond to irreducible representations V_λ.
The center of $U_q(\mathfrak{gl}_N)$ can be also identified (up to a certain completion) with symmetric functions via Drinfeld map:

$s_\lambda \mapsto$ 

Symmetric functions from quantum groups

Using the above identifications, we can define **Hopf pairing** on symmetric functions as follows:

$$\langle s_\lambda, s_\mu \rangle =$$



$$= s_\lambda(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}) s_\mu(q^{-N+1}, \ldots, 1).$$

This is symmetric in λ and μ.

Interpolation polynomials

Theorem (Knop,Okounkov,Olshansky,Sahi,...)

There exists a family of symmetric polynomials $F_\lambda(x_1,\ldots,x_N)$ such that

$$F_\lambda(q^{-\mu_1-N+1},\ldots,q^{-\mu_N})=0 \text{ unless } \lambda\subset\mu.$$

These polynomials are special cases of **interpolation Macdonald polynomials**. They diagonalize Hopf pairing:

$$\langle F_\lambda,F_\nu\rangle=\delta_{\lambda,\nu}q^{-|\lambda|+2\binom{N}{3}}\prod_{\square\in\lambda}(1-q^{N+c(\square)}),$$

where $c(\square)$ denotes the content of a box $\square$.

Interpolation polynomials

We can use polynomials F_λ to define the corresponding central elements σ_λ in $\mathcal{Z}(U_q(\mathfrak{gl}_N))$.

Theorem (Beliakova, G.)

- *For any knot K its universal invariant can be written as an infinite series*
$$J_K = \sum_\lambda a_\lambda(K)\,\sigma_\lambda$$
where a_λ are Laurent polynomials in q.
- *The coefficients a_λ are related to Reshetikhin-Turaev invariants $\mathrm{RT}(K; V_\lambda)$ by an explicit triangular matrix*
- *If q is a root of unity then all but finitely many terms in the above series vanish.*

This generalizes the results of Habiro for $U_q(\mathfrak{sl}_2)$.

Further directions

This is a very active area of research, and there are lots of open problems and conjectures. In particular:

- G., Neguț and Rasmussen conjectured a categorical interpretation of **Macdonald polynomials** and Bergeron-Garsia ∇ operator.
- Is it possible to draw other interesting symmetric functions using webs or links?
- How to categorify interpolation polynomials F_λ?

Thank You!