Braid varieties

Eugene Gorsky
University of California, Davis
joint with Roger Casals, Mikhail Gorsky, Ian Le,
Linhui Shen and José Simental

Yale Colloquium
December 7, 2022

Outline

For any positive braid β we define an affine algebraic variety $X(\beta)$ which we call **braid variety**. In this talk, we will discuss:

- Definitions of braid varieties
- Properties and examples
- Invariants and homology
- Cluster structure on braid varieties
- Braid varieties in all types.

Braid varieties

The braid group Br_n has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| \geq 2).$$

We define matrices

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & z & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

They satisfy the identity

$$B_i(z_1) B_{i+1}(z_2) B_i(z_3) = B_{i+1}(z_3) B_i(z_1 z_3 - z_2) B_{i+1}(z_1).$$

Braid varieties

Given a positive braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, we define the **braid variety**

$$X(\beta) = \{(z_1, \ldots, z_\ell) : w_0 B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell) \text{ is upper} - \text{triangular}\},$$

where

$$w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

It is an affine algebraic variety in $\mathbb{C}^\ell$. By the above, different braid words corresponding to the same braid define isomorphic varieties.

Braid varieties

As a warm-up example, consider the braid variety

$$X(\sigma^3) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_3 & -1 \\ 1 & 0 \end{pmatrix} \text{ upper-triangular} \right\} =$$

$$\{z_1 z_2 z_3 - z_1 - z_3 = 0\} = \{z_3(z_1 z_2 - 1) - z_1 = 0\}.$$

If $z_1 z_2 - 1 \neq 0$, then $z_3 = \frac{z_1}{z_1 z_2 - 1}$. If $z_1 z_2 - 1 = 0$ then $z_1 = 0$, contradiction. Therefore

$$X(\sigma^3) \simeq \{z_1 z_2 - 1 \neq 0\} \subset \mathbb{C}^2.$$

Note that this variety is **smooth** of dimension 2, and has a **holomorphic symplectic form**

$$\omega = \frac{dz_1 dz_2}{z_1 z_2 - 1}.$$

It is also an example of a **cluster variety** of type A_1 with one frozen variable.

Braid varieties

A more geometric definition of $X(\beta)$ uses the flag variety parametrizing complete flags

$$\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots F_n = \mathbb{C}^n : \dim F_i = i\}.$$

Two flags $\mathcal{F}, \mathcal{F}'$ are in position σ_i, if $F_j = F'_j$ for $j \neq i$, and $F_i \neq F'_i$. We will denote this as $\mathcal{F} \xrightarrow{\sigma_i} \mathcal{F}'$.

Given a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, we define the braid variety as the space of sequences of flags

$$X(\beta) = \left\{ \mathcal{F}^0 \xrightarrow{\sigma_{i_1}} \mathcal{F}^1 \xrightarrow{\sigma_{i_2}} \mathcal{F}_2 \cdots \xrightarrow{\sigma_{i_\ell}} \mathcal{F}^\ell = w_0 \mathcal{F}^0 \right\}$$

where $\mathcal{F}^0$ is a fixed standard flag.

This definition has a straightforward generaization for an arbitrary semisimple group G and the corresponding braid group Br_G.

Properties and examples

The braid varieties have the following properties:

- Two definitions of $X(\beta)$ are equivalent.
- $X(\beta)$ is either empty or it is a smooth affine algebraic variety of dimension $\ell(\beta) - \binom{n}{2}$.
- Any **open Richardson variety** $R^{\circ}_{w,u}$ can be realized as a braid variety.
- (Casals, G., Gorsky, Simental) Any **positroid variety** (defined by Knutson-Lam-Speyer) can be realized as a braid variety for several different braids.
- (Mellit) The character varieties for arbitrary Riemann surfaces can be stratified into braid varieties.
- (Kálmán,Casals-Ng) To any braid β one can associate a **Legendrian link** L_β such that the **augmentation variety** of L_β is isomorphic to $X(\beta)$. As a smooth link, L_β is the closure of $\beta\Delta^{-1}$ where Δ is the half-twist (positive braid lift of w_0).
- Shen-Weng considered a special class of braid varieties called **double Bott-Samelson cells**.

Properties and examples

As a concrete example, consider the braid

$$\beta_{n,m} = (\sigma_1 \cdots \sigma_{n-1})^m \Delta.$$

Then

$$X(\beta) \simeq \Pi_{n,m+n}/(\mathbb{C}^*)^m$$

where $\Pi_{n,m+n} \subset \mathrm{Gr}(n, m+n)$ is the **open positroid cell** in the Grassmannian defined by non-vanishing of cyclically consecutive $n \times n$ minors $\Delta_{i,i+1,\ldots,i+n-1}$. The link L_β is the (m, n) torus link.

Example

We have

$$X(\sigma^3) \times (\mathbb{C}^*)^2 = \Pi_{2,4} = \{\Delta_{1,2}\Delta_{2,3}\Delta_{3,4}\Delta_{1,4} \neq 0\} \subset \mathrm{Gr}(2,4).$$

This corresponds to the Hopf link $T(2,2)$.

Invariants and homology

- If β is reduced then $X(\beta)$ is a point (or empty).
- If β, β' are related by a braid move, $X(\beta) \simeq X(\beta')$.
- Let $\beta = \cdots\sigma_i\sigma_i\cdots, \beta' = \cdots\sigma_i\cdots$ and $\beta'' = \cdots 1 \cdots$ then there is an open embedding
$$X(\beta') \times \mathbb{C}^* \hookrightarrow X(\beta)$$
and the complement is isomorphic to $\mathbb{C} \times X(\beta'')$.

Theorem (Kálmán)

The number of points in $X(\beta)$ over a finite field $\mathbb{F}_q$ satisfies the recursion
$$\sharp X(\beta) = (q-1)\,\sharp X(\beta') + q\,\sharp X(\beta'')$$
Also, it is equal to the $(a=0)$ part of the **HOMFLY-PT polynomial** *of the link L_β.*

Invariants and homology

Since $X(\beta)$ is non-compact, its cohomology carries an interesting **weight filtration**.

Theorem (Trinh, Galashin-Lam)

The cohomology of $X(\beta)$ with weight filtration is isomorphic to the $(a=0)$ part of the triply graded **Khovanov-Rozansky homology** *of the link L_β.*

Theorem (Mellit)

The cohomology of $X(\beta)$ satisfies a "curious hard Lefshetz" property, in particular, it is a representation of $\mathfrak{sl}(2)$.

Theorem (G.,Hogancamp,Mellit)

The triply graded Khovanov-Rozansky homology of **any** *link satisfies a "curious hard Lefshetz" property, in particular, it has a certain symmetry conjectured by Gukov, Dunfield and Rasmussen.*

Invariants and homology

Example

This is the cohomology of the braid variety corresponding to the $(3, 4)$ torus knot:

	H^0	H^1	H^2	H^3	H^4	H^5	H^6
$k - p = 0$	1	0	1	0	1	0	1
$k - p = 1$	0	0	0	0	1	0	0

The rows correspond to the weight filtration on cohomology. The Lie algebra $\mathfrak{sl}_2$ acts in rows and there are two irreducible summands.

The total dimension of cohomology is $5 =$ Catalan number.

Cluster structures

Theorem (Casals, G., Gorsky, Le, Shen, Simental)

For any β the braid variety $X(\beta)$ is a **cluster variety***. In other words, the algebra of functions $\mathbb{C}[X(\beta)]$ is a* **cluster algebra***.*

- Parallel independent work by Galashin, Lam, Sherman-Bennett and Speyer.
- This resolves a conjecture of Leclerc on cluster structures on Richardson varieties.
- Leclerc's cluster structure was studied by Ménard, Keller–Cao and others. In special cases, conjecture has been proved by Serhiyenko–Sherman-Bennett–Williams and Galashin–Lam.

Cluster structures

Fock and Goncharov proposed a geometric approach to cluster algebras.

They introduced cluster varieties, coming in pairs (A, X), and stated certain **duality conjectures** for these. Under some assumptions, these conjectures were proved by Gross-Hacking-Keel-Kontsevich. The X-varieties allow for a natural quantization compatible with the Poisson bracket, while the A-varieties are a spectrum of a cluster algebra.

We show that braid varieties admit structures of both A and X cluster varieties, and the conditions used by GHKK are satisfied, so the Fock-Goncharov duality conjecture holds.

Note that the duality exchanges a semisimple group G with its Langlands dual group $G^\vee$ which share the same braid group. We prove that for a given braid β the braid varieties $X_G(\beta)$ and $X_{G^\vee}(\beta)$ form a dual pair.

Cluster structures

A (type A) cluster variety is an affine algebraic variety Y with the following structure:

- There is a collection of open charts $U \simeq (\mathbb{C}^*)^d$.
- Each chart U is equipped with **cluster coordinates** $A_1, \ldots, A_d$ which are invertible on U and extend to regular functions on Y. These coordinates could be either **mutable** or **frozen**.
- To each chart one assigns a skew-symmetric integer matrix ε_{ij} or a **quiver** with $\max(0, \varepsilon_{ij})$ arrows from vertex i to vertex j.
- For each chart U and each mutable variable A_k, there is another chart U' with cluster coordinates $A_1, \ldots, A'_k, \ldots, A_d$ and a skew symmetric matrix ε'_{ij} related by **mutation** μ_k.
- Any two charts in the collection are related by a sequence of mutations.
- The ring of functions on Y is generated by all cluster variables in all charts.

Cluster structures

In our running example $X(\sigma^3) = \{z_1 z_2 - 1 \neq 0\}$ we have two cluster charts:

- $U_1 = \{z_1 \neq 0, z_1 z_2 - 1 \neq 0\} \simeq (\mathbb{C}^*)^2$ with coordinates $z_1, z_1 z_2 - 1$
- $U_2 = \{z_2 \neq 0, z_1 z_2 - 1 \neq 0\} \simeq (\mathbb{C}^*)^2$ with coordinates $z_2, z_1 z_2 - 1$

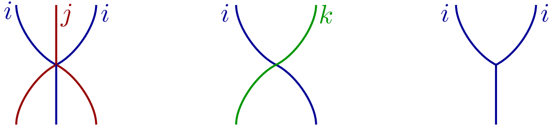
Note that

$$z_2 = \frac{(z_1 z_2 - 1) + 1}{z_1},$$

so the two charts are related by a mutation.

Algebraic weaves

To describe some of the cluster charts, we use the formalism of algebraic weaves (or simply weaves). These are graphs with edges labeled by σ_i which are built from elementary pieces

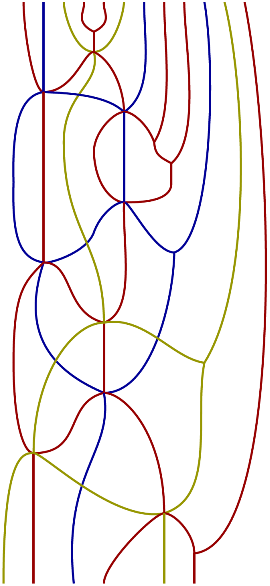


encoding braid moves and $\sigma_i\sigma_i \to \sigma_i$. Each horizontal section of a weave spells out a braid word, and we will always consider weaves with β on the top and w_0 on the bottom.

Theorem

Each algebraic weave defines an open chart in $X(\beta)$ isomorphic to $(\mathbb{C}^*)^d$ where $d = \dim X(\beta)$ is the number of trivalent vertices.

Algebraic weaves



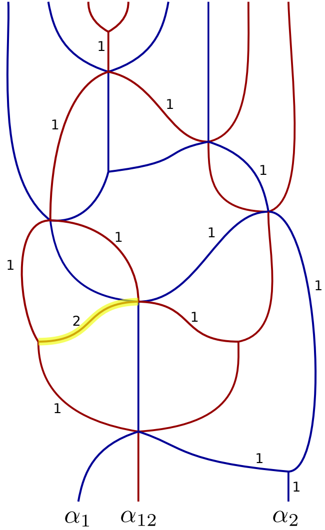
Algebraic weaves

Each weave corresponds to a "movie" of braids which sweeps out a surface Σ in $\mathbb{R}^4$. For a fixed weave:

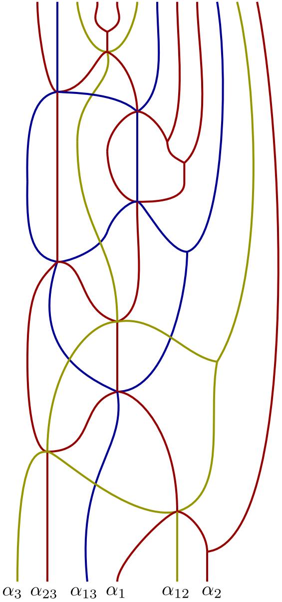
- We define a collection of cycles in $H_1(\Sigma, \partial\Sigma)$, which can be described combinatorially using linear combinations of edges in a weave.
- Each cycle starts at a trivalent vertex and propagates down according to certain rules. There is one cycle per trivalent vertex.
- The skew-symmetric matrix defining the quiver corresponds to the intersections between these cycles.
- Cycles which extend to the bottom correspond to frozen variables, other are mutable.

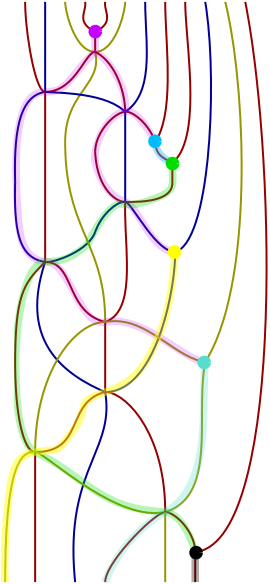
Algebraic weaves

Here is an example of the Lusztig cycle γ_v for the topmost vertex v:

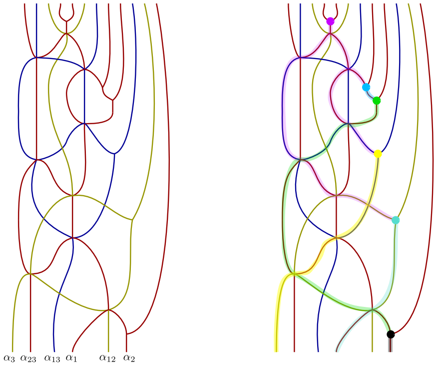


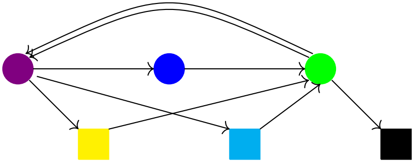
Algebraic weaves





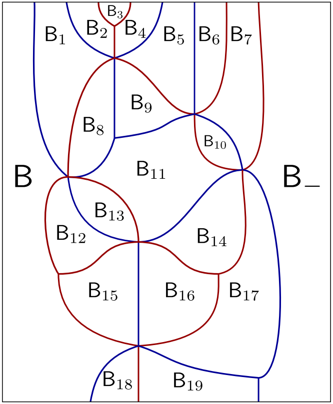
Algebraic weaves





Cluster coordinates

We can label all regions of a weave by complete flags in $\mathbb{C}^n$. If two flags are separated by an edge of color i, they are in position σ_i. The flags on the top encode a point in $X(\beta)$.



Cluster coordinates

Theorem

Given a weave $\mathfrak{W}$, there exists a unique collection of regular functions $A_v := A_v[z_1, \ldots, z_\ell]$, indexed by the trivalent vertices of the weave, and framed flags in the regions such that for every pair of regions r, r' separated by an edge e, the framed flags $B_r, B_{r'}$ are related by

$$B_i(\tilde{z})\chi_i\left(\prod_v A_v^{\gamma_v(e)}\right), \quad \text{for some } \tilde{z} \in \mathbb{C}.$$

Here $\chi_i(u) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & u & & & \\ & & & u^{-1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ and these A_v are **cluster variables**.

This completes the construction of cluster structure on $X(\beta)$.

Braid varieties in other types

The braid varieties can be defined for an arbitrary semisimple Lie group G, and the corresponding braid group Br_G. Many constructions and results can be generalized, in particular, cluster structures exist in full generality. More precisely:

- The point count on braid varieties is governed by the **Hecke algebra** while their homology is related to **Soergel bimodules** of the corresponding type.
- Weaves would look different, in particular, we will see $(2m_{ij})$-valent vertices in weaves.
- Although there are no links and surfaces, one can formally define the cycles and their intersections. We use pairs of **Lusztig cycles** (for a group G) and **dual Lusztig cycles** (for the Langlands dual group $G^\vee$) similar to coroots and roots.
- The cluster exchange matrix is only skew-symmetrizable, but not skew symmetric.

Thank You!