Torus knots and rational Cherednik algebras (joint with P. Etingof and I. Losev)

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Colored HOMFLY-PT invariants

Reshetikhin-Turaev invariants

Given the Lie algebra $\mathfrak{sl}_N$ and its representation V_λ, one can define the corresponding Reshetikhin-Turaev knot invariants. For a given knot K, the invariant $P^N_\lambda(K)$ is a polynomial in a single variable q.

For example, if K is the unknot, $P^N_\lambda(unknot)$ equals the q-character of V_λ and can be computed as:

$$P^N_\lambda(unknot) = s_\lambda(1, q, \dots, q^{N-1}),$$

where s_λ is the Schur polynomial labeled by λ.

The uncolored $\mathfrak{sl}_N$ invariants ($\lambda = \square$) can be defined using skein relations.

Colored HOMFLY-PT invariants

HOMFLY-PT invariants

It turns out that for a fixed Young diagram λ the N-dependence can be easily packed in a single function. Namely, there exists an invariant $P_\lambda(K)(a,q)$, called *colored HOMFLY-PT invariant*, such that

$$P_{\lambda,N}(K)(q) = P_\lambda(K)(a = q^N, q).$$

In the case of unknot, it can be computed as following. Remark that the power sums of $1, q, \ldots, q^{N-1}$ are equal to

$$p_k = 1 + q^k + \ldots + q^{(N-1)k} = \frac{1-q^{Nk}}{1-q^k} = \frac{1-a^k}{1-q^k}.$$

Therefore

$$P_\lambda(unknot) = s_\lambda\left(p_k = \frac{1-a^k}{1-q^k}\right).$$

Colored HOMFLY-PT invariants

Main theorem

Let $T(m,n)$ denote the (m,n) torus knot

Theorem (Etingof, G., Losev)

For all diagrams λ the following statements hold up to an overall scaling:
a) The function $P_{\lambda}(T(m,n))$ is a polynomial in $(-a)$ and a power series in q with **nonnegative coefficients**
b) If $d=|\lambda|$ then $P_{\lambda}(T(m,n))(1-q)\ldots(1-q^{d})$ is a polynomial in $(-a)$ and q with **nonnegative coefficients**.

The statement is false for $\mathfrak{sl}_N$ invariants, since the substitution $a=q^N$ is incompatible with the change of a to $(-a)$.

Colored HOMFLY-PT invariants

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Key idea: these coefficients compute the dimensions of certain pieces in certain representations of rational Cherednik algebra with parameter m/n.

Colored HOMFLY-PT invariants

Rosso-Jones formula

Colored invariants of torus knots were computed by Rosso and Jones:

- Raise all variables in the Schur polynomial s_λ to power n
- Expand the result in Schur basis:

$$s_\lambda(x_1^n, x_2^n, \ldots) = \sum_\mu c_{\lambda,n}^\mu s_\mu.$$

- The λ-colored invariant of $T(m,n)$ equals

$$P_\lambda(T(m,n)) = \sum_\mu q^{-\frac{m}{n}\kappa(\mu)} c_{\lambda,n}^\mu P_\mu(unknot),$$

where $\kappa(\mu)$ is the sum of contents of boxes in μ.

Rational Cherednik algebras

Definition

The rational Cherednik algebra associated with S_n is generated by $\mathbb{C}[S_n]$ and $2n$ additional generators $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ with the following relations:

$$[x_i, x_j] = 0,\ [y_i, y_j] = 0,\ [y_i, x_j] = c(i\,j),\ [y_i, x_i] = 1 - c\sum_{j \neq i}(i\,j).$$

It depends on the parameter c, which significantly affects the representation theory.

It has a polynomial representation $\mathbb{C}[x_1, \ldots, x_n]$:

- x_i act by multiplication
- S_n acts as usual
- y_i act by *Dunkl operators*:

$$y_i = \frac{\partial}{\partial x_i} - c\sum_{j \neq i} \frac{1 - (i\,j)}{x_i - x_j}.$$

Rational Cherednik algebras

Representations

More generally, if U_μ is a representation of S_n, then one can define

- "Highest weight" representation $M_c(\mu) = U_\mu \otimes \mathbb{C}[x_1, \ldots, x_n]$
- Its irreducible quotient $L_c(\mu)$

Each of them is a graded representation of S_n.We are ready to formulate more precise result:

Theorem

Let λ be a Young diagram of side d. The following identity holds:

$$P_\lambda(T(m,n)) = \sum_{i=0}^{n-1} (-a)^i \dim_q Hom_{S_{nd}} \left(\wedge^i W, L_{\frac{m}{n}}(n\lambda)\right),$$

where W is the reflection representation of S_{nd}.

Rational Cherednik algebras

Idea of proof

The proof is essentially a categorification of the Rosso-Jones formula and follows from two lemmas:

Lemma

The following identity holds in the Grothendieck group of representations:

$$[L_{\frac{m}{n}}(n\lambda)] = \sum_{\mu} c^{\mu}_{\lambda,n}[M_{\frac{m}{n}}(\mu)]$$

Lemma

The character of $M_{\frac{m}{n}}(\mu)$ can be computed by the formula:

$$\sum_{i=0}^{n-1}(-a)^i \dim_q Hom_{S_{nd}}\left(\wedge^i W, M_{\frac{m}{n}}(\mu)\right) = q^{-\frac{m}{n}\kappa(\mu)} P_{\mu}(unknot)$$

Rational Cherednik algebras

Example

Consider the rational Cherednik algebra of type S_2, without the loss of generality assume $x_1 + x_2 = y_1 + y_2 = 0$, and let $x = x_1 - x_2$. The polynomial representation is just $\mathbb{C}[x]$, and

$$y(f) = \frac{\partial f}{\partial x} - c\frac{f(x) - f(-x)}{x}.$$

Note that

$$y(x^{2k}) = 2kx^{2k-1},\ y(x^{2k+1}) = (2k+1-2c)x^{2k}.$$

Therefore for $c = 3/2$ we have $y(x^3) = 0$, so our algebra has a subrepresentation $x^3\mathbb{C}[x]$.
The corresponding irreducible quotient is 3-dimensional:

$$L_{\frac{3}{2}}(\square\square) = \langle 1, x, x^2\rangle.$$

This matches

$$P_{\square}(T(2,3)) = 1 - aq + q^2$$

Rational Cherednik algebras

Example cont'd

In the Rosso-Jones formula we start from $S_{\square} = \sum x_i$ and change it to

$$\sum x_i^2 = S_{\square\square} - S_{\substack{\square\\\square}}.$$

Therefore the invariant of $T(2,3)$ equals

$$P_{\square}(T(2,3)) = q^{-3/2} P_{\square\square}(unknot) - q^{3/2} P_{\substack{\square\\\square}}(unknot) =$$

$$q^{-3/2}\left(P_{\square\square}(unknot) - q^3 P_{\substack{\square\\\square}}(unknot)\right).$$

$P_{\square\square}(unknot)$ computes the character of $\mathbb{C}[x]$, and $P_{\substack{\square\\\square}}(unknot)$ computes the character of $x^3\mathbb{C}[x]$.

Khovanov and Rozansky defined knot homology theories categorifying $P_{\square}(a,q)$ and $P_{\square,N}(q)$ for all N.

Theorem (Rasmussen)

There exists a spectal sequence starting at HOMFLY-PT homology and converging to $\mathfrak{sl}_N$ homology.

Conjecture (G., Oblomkov, Rasmussen, Shende)

a) There exists a filtration on $L_{\frac{m}{n}}(n)$ such that the associated graded space is isomorphic to HOMFLY-PT homology of $T(m,n)$.
b) The differentials in the Rasmussen's spectral sequence can be explicitly expressed in terms of x_i and y_i.

Towards knot homology

Evidence for the conjecture

- The filtration is symmetric in m and n. From representation theory viewpoint, the isomorphism between certain components of $L_{\frac{m}{n}}(n)$ and $L_{\frac{n}{m}}(m)$ is very nontrivial.
- The filtration is symmetric in q and t. This symmetry corresponds to the involution on the Cherednik algebra, exchanging x_i and y_i.
- In "stable limit" $m \to \infty$ the filtration matches HOMFLY-PT homology.
- The answers match all known data for HOMFLY-PT and $\mathfrak{sl}_2$ homology.

Towards knot homology

Example: $(n, n+1)$ knots

For $m = n+1$ the filtration was constructed by Gordon, who proved that the associated graded is isomorphic to the space of *diagonal harmonics*:

$$grL_{\frac{n+1}{n}}(n) \simeq \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] / (\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{S_n}).$$

The character of this space was explicitly computed by Haiman in terms of Macdonald polynomials.

Thank you