Braid varieties and algebraic weaves

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Braid group: generators $\sigma_1 \dots \sigma_{n-1}$

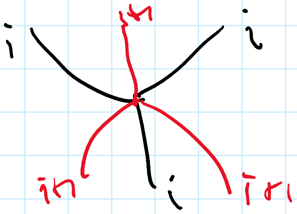
relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

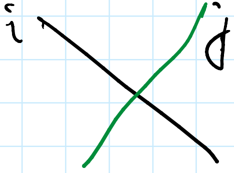
$\sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j|>1)$

① Diagrammatic category

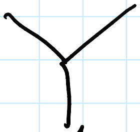
Objects: positive braid words (no σ_i^{-1})

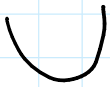
Morphisms: $\sigma_i \sigma_{i+1} \sigma_i \longleftrightarrow \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\qquad \sigma_i \sigma_j \longleftrightarrow \sigma_j \sigma_i$





$\sigma_i \sigma_i \longrightarrow \sigma_i$ $\qquad \sigma_i \sigma_i \longrightarrow 1$ + compositions thereof





- Lots of relations (later), similar to Soergel calculus but different in some ways.

Main result [CGGS] Realization of this category:

- Positive braid $\beta \rightsquigarrow$ braid variety $X_0(\beta)$
- Morphisms $\rightsquigarrow$ correspondences between braid varieties

. $X_0(\beta_0) \longleftarrow M \longrightarrow X_0(\beta_1)$

$$X_0(\beta_0) \leftarrow \mathcal{M} \longrightarrow X_0(\beta_1)$$

- Relations $\longrightarrow$ isomorphisms of correspondences

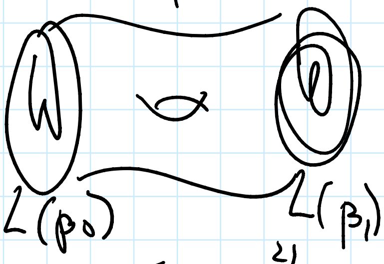
Composition

$$\begin{tikzcd} & & \mathcal{M} \times_{X_0(\beta_1)} \mathcal{M}' \arrow[dl] \arrow[dr] & & \\ & \mathcal{M} \arrow[dl] \arrow[dr] & & \mathcal{M}' \arrow[dl] \arrow[dr] & \\ X_0(\beta_0) & & X_0(\beta_1) & & X_0(\beta_2) \end{tikzcd}$$

Note [Casals-Zaslow] $\cdot$ $\beta \rightsquigarrow$ Legendrian link $L(\beta) \subset S^3$

- Morphism $\rightsquigarrow$ Lagrangian cobordism

(i.e. surface in $\mathbb{R}^4$, Lagrangian wrt to some symplectic structure)



$\Sigma \subset \mathbb{R}^4$

- Relations $\longrightarrow$ "Legendrian movie moves" $=$ Hamiltonian isotopies.

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② Braid matrices & braid varieties

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \underset{i\quad i+1}{\boxed{\begin{matrix} 0 & 1 \\ 1 & z \end{matrix}}} & & \\ & & & & 1 & \\ & & & & & \ddots \; 1 \end{pmatrix}$$

$\overset{\shortparallel}{n \times n}$ matrix

$\beta = \sigma_{i_1} \cdots\cdots \sigma_{i_r}$

$$B_\beta(z_1 \ldots z_r) = B_{i_1}(z_1) \ldots B_{i_r}(z_r)$$

braid matrix

$$X_0(\beta) = \{ (z_1 \ldots z_r) \mid B_\beta(z_1 \ldots z_r) \text{ is upper-triangular} \} \subset \mathbb{C}^r$$

affine alg. variety

(Mellit).

$$\boxed{B_i(z_1)\, B_{i+1}(z_2)\, B_i(z_3) = B_{i+1}(z_3)\, B_i(z_2 - z_1 z_3)\, B_{i+1}(z_1)}$$

$$(z_1, z_2, z_3) \longleftrightarrow (z_3, z_2 - z_1 z_3, z_1)$$

$$(z_1, z_2, z_3) \longleftrightarrow (z_3, z_2 - z_1 z_3, z_1)$$

equivalent braids $\longleftrightarrow$ isomorphic braid varieties.

$$X_0(\beta, \pi) = \{ B_\beta(z_1 \ldots z_n) \cdot \pi \text{ upper triangular} \}$$

$\pi \in S_n$

Ex: $\beta = \sigma_1^4$ $\begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix}$

$$= \begin{pmatrix} * & * \\ z_1 + z_3 + z_1 z_2 z_3 & * \end{pmatrix}$$

$$X_0(\sigma_1^4) = \{ (z_1, z_2, z_3, z_4) : z_1 + z_3 + z_1 z_2 z_3 = 0 \}$$

$z_1 + z_3(1 + z_1 z_2) = 0$ $\quad$ $1 + z_1 z_2 = 0 \Rightarrow z_1 = 0$ contradiction

$1 + z_1 z_2 \neq 0 \Rightarrow z_3 = \dfrac{-z_1}{1 + z_1 z_2}$

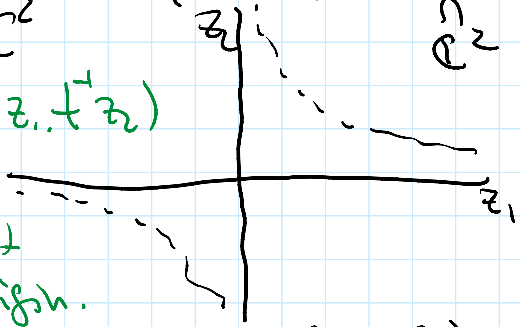
Conclusion: $X_0(\sigma_1^4) = \{ 1 + z_1 z_2 \neq 0 \} \times \mathbb{C}_{z_4}$

$\cap$
$\mathbb{C}^2$

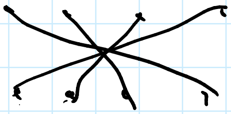
$\mathbb{C}^2$

$(z_1, z_2) \rightsquigarrow (t z_1, t^{-1} z_2)$

$\mathbb{C}^*$ not free, fixed pt at the origin.



$W_0 = \Delta =$ half twist $\left(\begin{matrix} n = 2 \\ \Delta = \sigma_1 \end{matrix} \right)$



$\Delta^2 =$ full twist

Thm [CGGS] (a) $X_0(\beta \Delta^2) \simeq X_0(\beta \Delta, W_0) \times \mathbb{C}^{\binom{n}{2}}$ $\leftarrow z_4$

(b) $X_0(\beta \Delta, W_0)$ smooth, $\dim = \ell(\beta) =$ "expected dimension"

$\ell(\beta) + \binom{n}{2}$ variables

$\binom{n}{2}$ equations $\longrightarrow$ upper-triangular

(c) There's a free action of some torus T

(c) There's a *free* action of some torus T on $X_0(\beta\Delta, w_0)$ such that $X_0(\beta\Delta, w_0)/T$ smooth, holomorphic symplectic

(d) $X_0(\beta\Delta, w_0)/T \simeq Aug(\beta)$ augmentation variety for Chekanov-Eliashberg DGA.

(Kálmán)

Definition of T depends on # components in the closure of β. If β closes to a knot, $T = (\mathbb{C}^*)^{n-1}$

Also: $(\mathbb{C}^*)^{n-1}$ acts on $X_0(\beta\Delta, w_0)$, but the action is not free in general.

(e) $X_0(\gamma, w_0)$ is a complete intersection, irreducible *or* empty

γ contains w_0 as a subword

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③ Morphisms and correspondences

Braid relations: $B_i(z_1)B_{i+1}(z_2)B_i(z_3)$
$= B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$

$B_i(z_1)B_j(z_2) = B_j(z_2)B_i(z_1)$

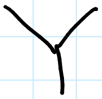
→ isomorphisms of braid varieties

Lemma $B_i(z)\underset{\uparrow}{U} = \underset{\uparrow}{U'}B_i(z')$

Lemma $B_i(z)U = U' B_i(z')$

upper-triag

$\sigma_i\sigma_i \to \sigma_i$



$$\cdots B_i(z_1)B_i(z_2)\cdots$$
$$\|$$
$$\cdots\begin{pmatrix}0&1\\1&z_1\end{pmatrix}\begin{pmatrix}0&1\\1&z_2\end{pmatrix}\cdots$$
$$\|$$
$$\cdots\underbrace{\begin{pmatrix}-z_1^{-1}&1\\0&z_1\end{pmatrix}}\begin{pmatrix}0&1\\1&z_2+z_1^{-1}\end{pmatrix}\cdots$$

some change of vars.

$$U'\cdots\cdots B_i(z_2+z_1^{-1})\cdots\cdots$$

$$X_0(\cdots\sigma_i\sigma_i\cdots)\leftarrow\mathcal{M}\longrightarrow X_0(\cdots\sigma_i\cdots)$$

injective
image $=\{z_1\neq 0\}$

trivial $\mathbb{C}^*$ fibration

$\sigma_i\sigma_i \longrightarrow \mathbb{1}$

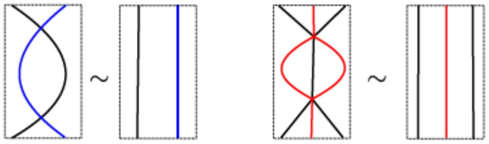
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&z\end{pmatrix}=\begin{pmatrix}1&z\\0&1\end{pmatrix}$$

upper-triag.

$$X_0(\cdots\sigma_i\sigma_i\cdots)\leftarrow\mathcal{M}\longrightarrow X_0(\cdots)$$

inj
image $=\{z_1=0\}$

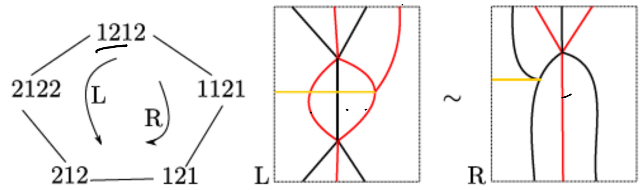
trivial $\mathbb{C}$ fibration



④ Relations

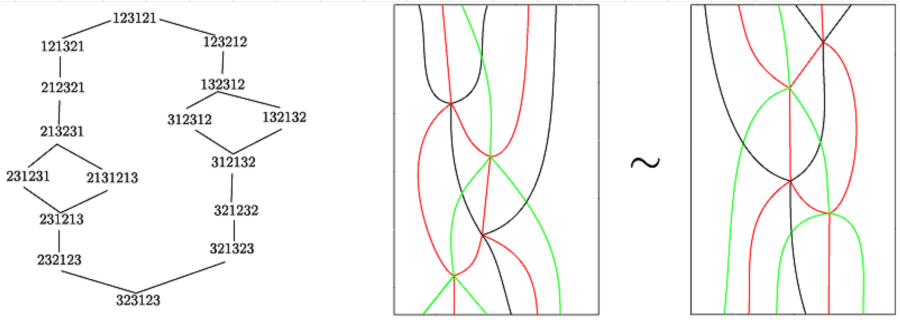
braid moves are invertible

$$\begin{tikzcd}\sigma_1\sigma_2\sigma_1\sigma_2 \arrow[d] \\ \sigma_1\sigma_2\sigma_1\end{tikzcd}$$



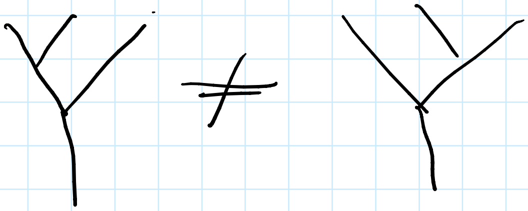
$\sigma_1\sigma_2\sigma_1$

Two ways to get from $\sigma_1\sigma_2\sigma_1\sigma_2 \to \sigma_1\sigma_2\sigma_1$ give isomorphic correspondences



Zamolodchikov relation.

But:



"mutation"

Application

$$\begin{tikzcd}
\beta_0 \arrow[d, dash, dotted] & & \mathcal{M} \arrow[dl] \arrow[dr] & \\
\beta_1 & X_0(\beta_0) & & X_0(\beta_1)
\end{tikzcd}$$

$$\mathcal{M} \longrightarrow X_0(\beta_0) \text{ injective}$$

$$\mathcal{M} \simeq (\mathbb{C}^*)^{\#3v} \times (\mathbb{C})^{\#\text{cups}} \times X_0(\beta_1)$$

Thm (a) There is a family of toric charts $(\mathbb{C}^*)^{\ell(\beta)}$ covering $X_0(\beta\Delta, w_0)$ up to codimension 2, these correspond to morphisms with no cups.

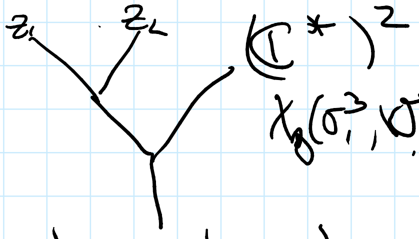
(b) The complements to these charts can be stratified by

$$\mathbb{C}^a \times (\mathbb{C}^*)^b \qquad 2a+b = \ell(\beta)$$

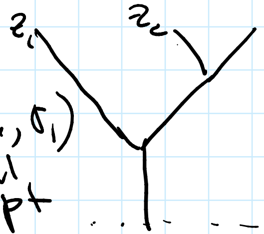
$$\mathbb{C}^a \times (\mathbb{C}^*)^b \qquad 2a+b = \ell(\beta)$$

Ex $\beta = \sigma_1^2 \quad \beta\Delta = \sigma_1^3 \quad X_0(\beta\Delta, w_0) = \{z_1z_2+1 \neq 0\} \subset \mathbb{C}^2$

2 charts: $\{z_1\neq 0, z_1z_2+1\neq 0\}$ $\qquad$ $\{z_2\neq 0, z_1z_2+1 \neq 0\}$



$$\begin{tikzcd} X_0(\sigma_1^3, \sigma_1) & \mathcal{M} \arrow[l] \arrow[r] & X_0(\sigma_1,\sigma_1) \arrow[d, phantom, "\parallel"] \\ & & \mathrm{pt} \end{tikzcd}$$



Complement $= \{z_1 = 0\}$ $\qquad \mathbb{C} \qquad$ $\{z_2=0\}$

Gao/Shen/Weng: cluster structure on $X_0(\beta)$

Conj: these toric charts are cluster charts.

SBim_n $\qquad T_iT_{i+1}T_i \longleftrightarrow T_{i+1}T_iT_{i+1}$

$$T_iT_j \longleftrightarrow T_jT_i$$

$$[T_i^2 \longrightarrow \mathbb{1}] \simeq \underbrace{[T_i \longrightarrow T_i]}_{\text{"skein exact triangle"}}$$

$$(T\to T) \dashrightarrow T^2$$

"Open Bott-Samelson variety"

$$\begin{tikzcd} & \sigma_{i_1} & & \cdots & & \sigma_{i_r} & \\ \underline{\underline{\mathcal{F}_0}} \arrow[ur, bend left] & & \mathcal{F}_1 \arrow[ul, bend right] & \cdots & \mathcal{F}_{r-1} \arrow[ur, bend right] & & \underline{\underline{\mathcal{F}_r}} \arrow[ul, bend right] \end{tikzcd}$$

Broué-Michel

Deligne

Shende-Treumann-Zaslow.

$\mathcal{F}_i$ = flag $\qquad$ $\mathcal{F}_j$ and $\mathcal{F}_{j+1}$ are the same except for

Closure:

Closure:
first and last
flags are same.

same except for
i_j-th place
different at i_j-th plac

$$\big(G \times X_0(\beta)\big)/B = \text{open BS.}$$

$$B_\Delta\left(z_1 \dots z_{\binom{n}{2}}\right) = \begin{pmatrix} 0 & & & 1 \\ & & \iddots & z_1 \\ & \iddots & & z_2\, z_3 \\ \Delta & & & \vdots \\ & \ldots & \ldots & \vdots \end{pmatrix}$$

$$B_{\Delta^2} = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & * \end{pmatrix}\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & * \end{pmatrix} = \underline{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$$