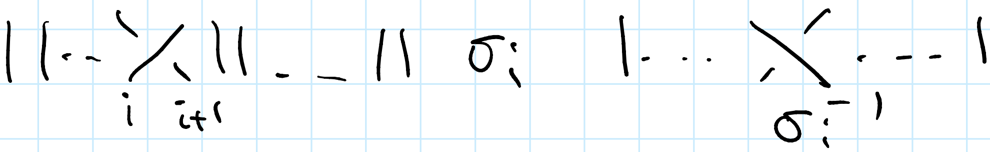
Braids, matrices and character varieties (after Mellit)

Thursday, October 15, 2020 8:57 AM

Anton Mellit "Cell decompositions of character varieties" https://arxiv.org/abs/1905.10685

Braid group generators $\sigma_1 \ldots \sigma_{n-1}$



Relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1$

We will be interested in positive braids $\Longrightarrow$ no σ_i^{-1}

Matrix $B_i(z) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 1 & z \end{matrix}} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ i-th and i+1 position

Given a braid

$\beta = \sigma_{i_1} \ldots \sigma_{i_k}$ = braid word in generators σ_i

$\Rightarrow B(\beta) = B_{i_1}(z_1) B_{i_2}(z_2) \ldots B_{i_k}(z_k)$

braid matrix

$z_1 \ldots z_k$ = variables

Variety $\boxed{M(\beta) = \{(z_1 \ldots z_k) \mid B(\beta) \text{ is upper triangular}\}}$

variety of β) $= \{(z_1 \ldots z_k) \mid B(\beta)$ is upper triangular$\}$.

Lemma If we write β differently, we get an isomorphic variety

Proof: if $|i-j|>1$ then clearly $B_i(z)B_j(z') = B_j(z')B_i(z)$

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$B_i(z_1)B_{i+1}(z_2)B_i(z_3) =$

$$\begin{array}{c} i \\ i+1 \\ i+2 \end{array}\begin{pmatrix} 0 & 1 & 0 \\ 1 & z_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & z_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & z_1 \\ 0 & 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & z_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & z_1 \\ 1 & z_3 & z_2 \end{pmatrix} = (\text{check}) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$$

Change of variables $(z_1, z_2, z_3) \to (z_3, z_2 - z_1z_3, z_1)$

$$\mathcal{M}(\ldots \sigma_i\sigma_{i+1}\sigma_i \ldots) \simeq \mathcal{M}(\ldots \sigma_{i+1}\sigma_i\sigma_{i+1} \ldots)$$

$\square$

Example  $= \sigma_1^4$

$B(\beta) = B_1(z_1)B_1(z_2)B_1(z_3)B_1(z_4) =$

$$= \begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ z_1 & 1+z_1z_2 \end{pmatrix}\begin{pmatrix} 1 & z_4 \\ z_3 & 1+z_3z_4 \end{pmatrix}$$

$\begin{pmatrix} * & * \\ & \end{pmatrix}$ $B(\beta)$ upper-triangular

$$= \begin{pmatrix} * & * \\ z_1 + z_3(1+z_1 z_2) & * \end{pmatrix}$$

$B(\beta)$ upper-triangular $\iff \boxed{z_1 + z_3(1+z_1 z_2) = 0}$

z_4 = arbitrary

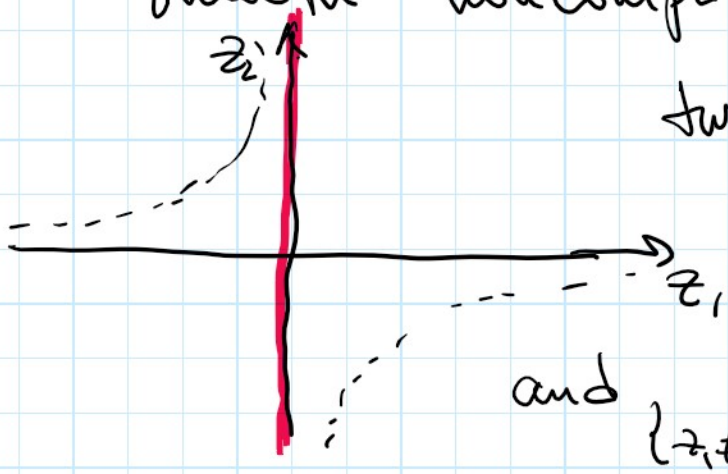
If $1 + z_1 z_2 = 0$

then $z_1 = 0$, contradiction

So $1 + z_1 z_2 \neq 0$ and $z_3 = \dfrac{z_1}{1+z_1 z_2}$

$\mathcal{M}(\beta) = \{1 + z_1 z_2 \neq 0\} \times \mathbb{C}_{z_4}$

Smooth noncompact variety, dim $= 3$.



two strata:

$\{z_1 = 0\}$, z_2 arbitrary

$\mathbb{C}_{z_2} \times \mathbb{C}_{z_4}$

and

$\{z_1 \neq 0, \ 1 + z_1 z_2 \neq 0\} = \mathbb{C}^* \times \mathbb{C}^*$

$\mathbb{C}^*_{z_1} \times \mathbb{C}^*_{1+z_1 z_2} \times \mathbb{C}_{z_4}$

In particular,

$$[\mathcal{M}(\sigma_1^4)] = \mathbb{L}^2 + (\mathbb{L}-1)^2 \mathbb{L} = \mathbb{L}^2 + \mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}$$
$$= \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L}.$$

In fact, we can check that $H_*(\mathcal{M}(\sigma_1^4))$ is 3-dimensional, but the weight filtration is 3-dimensional, but the weight filtration is interesting.

Alexander duality between $H_*(\{1+z_1z_2=0\})$ and $\tilde{H}^{\text{compact}}_*(\{1+z_1z_2\neq 0\})$

$$\{1+z_1z_2=0\} \cong \mathbb{C}^*$$

H_0, H_1 $\qquad\qquad$ $H_2, H_3 + H_0$

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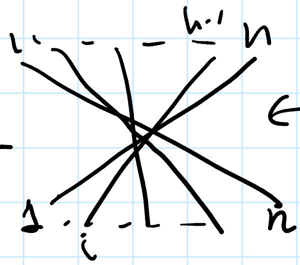
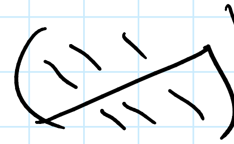
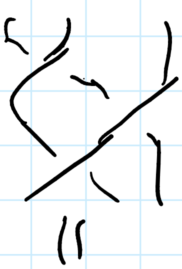
① Half-twist $\qquad\qquad$ Full twist

$\Delta =$  ← all positive crossing

$\Delta^2 = \Big($  $\Big)^n$

<u>Observation</u>: $B(\Delta) = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & z_{ij} \end{pmatrix}$ ← all independent variables.

$n=3$

$\Delta =$  $= \sigma_1\sigma_2\sigma_1$

There are $\binom{n}{2}$ crossing in Δ

$\longleftrightarrow$ $\binom{n}{2}$ variables under anti-diagonal.

$$B(\sigma_1\sigma_2\sigma_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & z_1 \\ 1 & z_3 & z_2 \end{pmatrix}$$

Can write $B(\Delta) = L \cdot \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \cdot U$

lower triangular

upper triangular.

So $B(\Delta^2) = B(\Delta) \cdot B(\Delta) =$

two different sets of variables

$$= \underbrace{L \cdot \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}}_{B(\Delta)} \cdot \underbrace{\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \cdot U}_{B(\Delta)} = LU$$

independent lower/upper triangular matrices with 1's on diagonal.

<u>Note</u> $B(\Delta)$ is never upper-triangular!

$$\boxed{\mathcal{M}(\Delta) = \emptyset}$$

$B(\Delta^2)$ is upper-triangular $\Leftrightarrow L = I$

$$\Rightarrow \boxed{\mathcal{M}(\Delta^2) = \mathbb{C}^{\binom{n}{2}}}$$

② Suppose that the braid contains Δ^2:

$B(\beta \cdot \Delta^2) = B(\beta) \cdot B(\Delta^2) =$

$= B(\beta) \cdot LU$

$= B(\beta) \cdot L\tilde{U}$

This is upper triangular, if $B(\beta) \cdot L$ upper triangular

$B(\beta) \cdot LU = U'$

$B(\beta) = U' U^{-1} L^{-1} \iff B(\beta)^{-1} = L \cdot U''$

$B(\beta)^{-1}$ has an LU decomposition

This is an open condition (principal minors $\neq 0$).

Thus $\mathcal{M}(\beta \cdot \Delta^2) = \left\{ \begin{array}{l} \text{open subset} \\ \text{where } B(\beta)^{-1} \\ \text{has LU decomp} \end{array} \right\} \times \mathbb{C}_{U}^{\binom{n}{2}}$

variables in U are arbitrary.

Ex $\sigma_1^4 = (\sigma_1^2) \cdot \sigma_1^2$, where $\sigma_1^2 = \beta$ and $\sigma_1^2 = \Delta^2$. z_4 does not matter

In particular, $\mathcal{M}(\beta \cdot \Delta^2)$ is non-empty, smooth of expected dimension ($= \#$ crossings in $\beta + \binom{n}{2}$) and irreducible.

③ How to compute $[\mathcal{M}(\beta)]$ in the Grothendieck ring of varieties?

varieties.

What happens if $\beta = \ldots \sigma_i \sigma_i \ldots$

$$B_i(z_1) B_i(z_2) = \begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ z_1 & 1 + z_1 z_2 \end{pmatrix}$$

(a) $z_1 \neq 0$ we can write it as

$$B_i(z_1) B_i(z_2) = \begin{pmatrix} -z_1^{-1} & 1 \\ 0 & z_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_2 + z_1^{-1} \end{pmatrix}$$

upper triang.

(b) $z_1 = 0$ we can write

$$B_i(0) B_i(z_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix}$$

$B(\beta) = \ldots B_i(z_1) B_i(z_2) \ldots =$

(a) $\ldots U \cdot B_i(z_2 + z_1^{-1}) \ldots$

(b) $\ldots U \ldots$

upper triangular

<u>Useful lemma</u> We can push the upper triangular matrices to the left.

$$B_i(z) \cdot U = \tilde{U} B_i(z')$$

$$\begin{pmatrix} 0 & 1 \\ & \end{pmatrix} \begin{pmatrix} a & b \\ & \end{pmatrix} = \begin{pmatrix} c & 0 \\ & \end{pmatrix} \begin{pmatrix} 0 & 1 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z' \end{pmatrix}$$

$$\begin{pmatrix} 0 & c \\ a & b+cz \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & c \\ a & az' \end{pmatrix}$$

$$z' = \frac{1}{a}(b+cz).$$

In the above computation, can keep pushin U to the left and changing z-variables

(a) $\cdots\cdots U B_i(z_2+z_1^{-1})\cdots = \tilde{U} \underbrace{\cdots\cdots}_{\text{some change of variables}} B_i(z_2+z_1^{-1})\cdots$

(b) $\cdots\cdots U \cdots\cdots = \tilde{U}\cdots\cdots$

<u>Thm</u> We can stratify $\mathcal{M}(\cdots\sigma_i\sigma_i\cdots)$ as follows: (a) $\mathcal{M}(\cdots\sigma_i\cdots)\times\mathbb{C}^*_{z_1}$ $(z_1\neq 0)$

(b) $\mathcal{M}(\cdots 1 \cdots)\times\mathbb{C}_{z_2}$ $(z_1=0)$

<u>Proof</u> (a) $B(p) = \cdots\cdots UB_i(z_2+z_1^{-1})\cdots =$

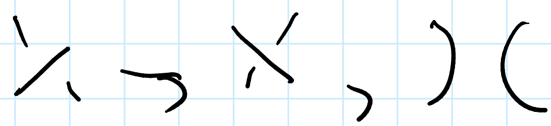
$= \tilde{U}\cdots\cdots B_i(z_2+z_1^{-1})\cdots$

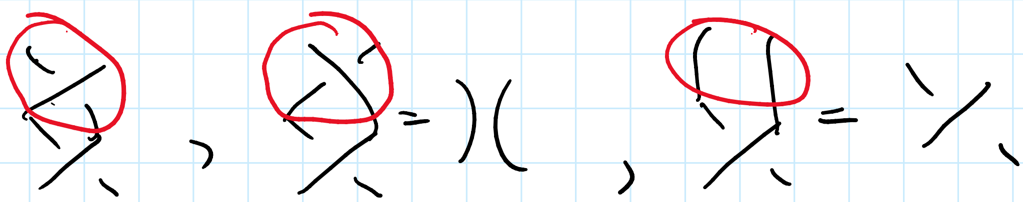
This is upper-triangular if $\cdots B_i(z_2+z_1^{-1})\cdots$ is upper-triangular, and we can is upper-triangular, and we can reconstruct z_2 from $z_2+z_1^{-1}$ and z_1.

Cor $[\mathcal{M}(\cdots\sigma_i\sigma_i\cdots)] =$

$$= (\mathbb{L}-1)\,[\mathcal{M}(\cdots\sigma_i\cdots)] + \mathbb{L}\,[\mathcal{M}(\cdots 1 \cdots)]$$

Note: This is similar to skein relation!

 ← linear relation



$\sigma_i\sigma_i$ $\qquad 1 \qquad$ σ_i

$$\langle \sigma_i\sigma_i \rangle = (\mathbb{L}-1)\,\sigma_i + \mathbb{L}\,1.$$

We can compute it recursively and come to expressions which do not contains squares after any application of braid relations.

Fact: Such expressions are in bijection with permutations S_n

bijection with permutations S_n.

$w \in S_n \longrightarrow$ "positive lift"

$w(i)$

i

connect i and $w(i)$ and draw all crossings positively.

$$\underbrace{\sigma_1 \sigma_2 \sigma_1 \sigma_2}_{\text{contains a square after applying braid relation.}} = \sigma_1 \sigma_1 \sigma_2 \sigma_1$$

$$Br_n \longrightarrow S_n = \langle \text{braid relation}, \ s_i^2 = 1 \rangle.$$

<u>Observation</u> $w = 1$ then $B(1) = I$

$\mathcal{M}(1) = pt$.

$\underset{S_n}{w} \neq 1 \Rightarrow B(w)$ has 1's corresponding to the permutation matrix of w and z_{ij} somewhere else (as for Δ).

This is <u>never</u> upper-triangular, and

$B(w) = \emptyset$.

<u>Thm</u> $[\mathcal{M}(\beta)]$ is always a polynomial

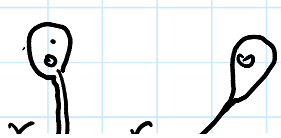
__Thm__ $[\mathcal{M}(\beta)]$ is always a polynomial in $\mathbb{L}$ (which is related to HOMFLY polynomial of β).
(lowest degree in a).

follows from skein rels.

What else is known about it?

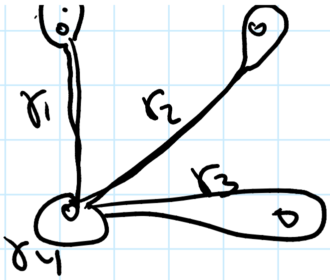
- (Mellit) There is a holomorphic symplectic form ω on $\mathcal{M}(\beta\cdot\Delta^2)$ and multiplication by ω satisfies "curious hard Lefshetz" property with respect to weight filtration.
  $\rightsquigarrow$ symmetry of dimension in $Gr_W H_*$
- (Mellit) Character varieties for surfaces can be decomposed into strata similar to $\mathcal{M}(\beta)$

__Ex__ $\mathbb{P}^1$, 4 punctures



represes $\pi_1(\mathbb{P}^1 - 4\,pts)$

$\longrightarrow GL(2)$



represent. $\pi_1(\mathbb{P}^1 - 4\text{pts})$

$\longrightarrow GL(2)$

fixing conjugacy classes around the puncture.

$\rho(\gamma_i)$ are conjugate to generic diagonal matrices.

$$\rho(\gamma_i) = \alpha_i C_i \alpha_i^{-1}$$

C_i diagonal

$$\boxed{\alpha_1 C_1 \alpha_1^{-1} \cdot \alpha_2 C_2 \alpha_2^{-1} \cdot \alpha_3 C_3 \alpha_3^{-1} = C_4} \quad (*)$$

C_1, C_2, C_3 fixed, but $\alpha_1, \alpha_2, \alpha_3$ can vary.

<u>Idea:</u> Can look at differen space

$$\alpha_1 C_1 \alpha_1^{-1} \alpha_2 C_2 \alpha_2^{-1} \alpha_3 C_3 \alpha_3^{-1} = \{\text{upper trian. with } C_4 \text{ on diag.}\}$$

This is affine fibration over $(*)$

This can be presented as a union of $\mathcal{M}(\beta)$ for $\beta = \sigma_1^2, \sigma_1^4, \sigma_1^6$.

- (Kálmán)

Given a braid β, Chekanov defined

Given a braid β, Chekanov defined a differential graded algebra $\mathcal{A}_\beta$

$\mathcal{M}(\beta) \simeq$ Augmentation variety for $\mathcal{A}_\beta$, that is, the space of homomorphisms $\{\mathcal{A}_\beta \to \mathbb{C}\}$

Kálmán: $B(\beta)\cdot L$ is upper triangular

$\wr\wr$ $\quad$ Lower-triangular.

$B(\beta)\cdot L\cdot U = B(\beta\cdot\Delta^2)$