Knot invariants and interpolation Macdonald polynomials

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Top. properties of knot invariants (and link) $\longleftrightarrow$ center of $U_q\mathfrak{gl}_N$ $\longleftrightarrow$ combinatorics of symmetric functions

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Reps of $U_q\mathfrak{gl}_N$ (finite-dimensional type I) $\longleftrightarrow$ reps of $\mathfrak{gl}_N$ $\longleftrightarrow$ reps of $GL(N)$

K_i act by powers of q

labeled by partitions

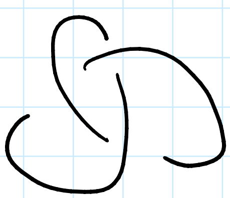
$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$

with at most N parts

V_λ = irreducible rep. of $U_q\mathfrak{gl}_N$

character of V_λ = Schur polynomial $s_\lambda(x_1, \dots x_N)$.

Reshetikhin–Turaev invariants



K = knot (oriented, framed)

and a representation V_λ of $U_q\mathfrak{gl}_N$

$\rightsquigarrow$ RT invariant $J_\lambda(K) \in \mathbb{Z}[q, q^{-1}]$

topological invariant of K

$\underline{\underline{Ex}}$ $\lambda = \square = (1, 0)$ $N = 2$

$J_\square(K)$ = Jones polynomial of K

L = link with r components (oriented, framed)

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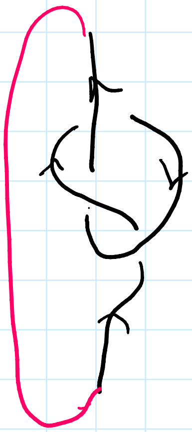
$V_{\lambda^{(1)}}, \dots, V_{\lambda^{(r)}}$ = r-tuple of reps of $U_q gl_N$

$\rightsquigarrow J_{\lambda^{(1)} \dots \lambda^{(r)}}(L)$ = top. invariant of link.

Cut a knot open at one place and get (1,1) tangle



<u>universal</u> RT invariant

$$\boxed{J(K) \in \widehat{\mathcal{Z}}(U_q gl_N)}$$

some completion of the center of $U_q gl_N$

If we choose representation V_λ and close the knot, then

$$J_\lambda(\bar{K}) = Tr_q\left(J(K)\big|_{V_\lambda}\right)$$

restrict the element of the center to the rep. V_λ

$J(K)$ can be constructed using universal R-matrices for $U_q gl_N$.

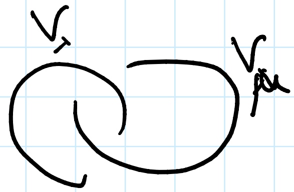
<u>Conclusion</u> If we know $J(K) \in \widehat{\mathcal{Z}}(U_q gl_N)$ we can compute all RT invariants of K.

Conversely, we can reconstruct $J(K)$ from all its traces in all $V \Leftrightarrow$ all RT invariants of K

all its traces in all V_λ $\Longleftrightarrow$ all RT invariants of K.

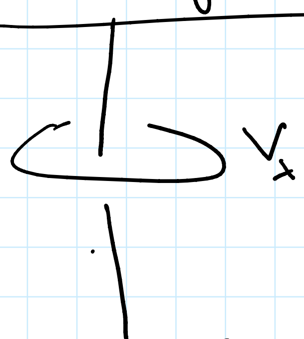
Rmk Because V_λ is irreducible, and $J(K)$ central, $J(K)$ acts on V_λ by a scalar

$$Tr\left(J(K)|_{V_\lambda}\right) = \Big(\underbrace{\text{this scalar}}_{J(K)|_{V_\lambda}}\Big) \cdot \underbrace{Tr_q\left(1|_{V_\lambda}\right)}_{\text{known.}}$$

Example  RT invariant of Hopf link

Hopf link $= s_\lambda(q^{-\mu_1-N+1}, q^{-\mu_2-N+2}, \dots, q^{-\mu_N}) \cdot s_\mu(q^{-N+1}, q^{-N+2}, \dots, q^0)$ $\Big\}$ symmetric in λ and μ.

S_λ, S_μ = Schur polynomials
= symmetric functions in N vars.



$\Rightarrow$ central element in $\widehat{Z(U_q\mathfrak{gl}_N)}$ which acts in V_μ by a scalar $s_\lambda(q^{-\mu_1-N+1}, \dots, q^{-\mu_N})$.

Check  $\rightsquigarrow$ trace in V_μ

$$= s_\lambda(q^{-\mu_1-N+1}, \dots, q^{-\mu_N}) \cdot Tr_q(1|_{V_\mu})$$

$$\qquad\qquad \shortparallel$$
$$s_\mu(q^{-N+1}, \dots, 1)$$

Drinfeld: One can construct lots of central elements this way by choosing different V_λ rep.

elements this way by choosing different V_λ rep.

or a formal linear combination of reps.

Thm (Drinfeld) These elements span the center (up to completion)

so $\widehat{Z(U_q\mathfrak{gl}_N)} = \left(\begin{smallmatrix}\text{completion}\\ \text{of}\end{smallmatrix}\right)\mathbb{C}[x_1 \dots x_N]^{S_N}$

$\bigcirc\, V_\lambda \longleftarrow S_\lambda$

Thm (Beliakova–G.) (a) There is a basis F_λ in $\widehat{Z(U_q\mathfrak{gl}_N)}$ such that

$F_\lambda|_{V_\mu} = 0$ for μ not containing λ

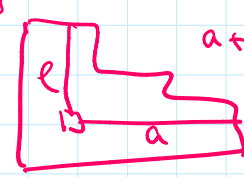
$\lambda \leftrightarrow$ Young diagram

some explicit power

hook length of a box in λ

$$F_\lambda|_{V_\lambda} = \pm q^{\cdots} \prod_{\square\in\lambda}\left(1-q^{h(\square)}\right)$$

(b) For any 0-framed knot K,



$a+l+1 = h(\square)$.

we can expand $\overbrace{\text{RT invariant}}^{\text{universal}}$ in this basis

$$J(K) = \sum_\lambda a_\lambda F_\lambda \quad \text{where } a_\lambda \in \mathbb{Z}[q,q^{-1}]$$

Cor: Knowing these coefficients a_λ, we can compute all RT invariants of K as following:

$J_\mu(K) = Tr(J(K)|_{V_\mu}) = \sum a_\lambda Tr(F_\lambda|_{V_\mu}) =$

$$J_\mu(K) = \mathrm{Tr}\left(J(K)\big|_{V_\mu}\right) = \sum_\lambda a_\lambda \mathrm{Tr}\left(F_\lambda\big|_{V_\mu}\right) =$$

$$= \dim_q V_\mu \cdot \sum_\lambda a_\lambda \left(F_\lambda\big|_{V_\mu}\right)$$ know

Matrix $C = (c_{\lambda,\mu}) = \left(F_\lambda\big|_{V_\mu}\right)$ is triangular

with known elements $F_\lambda\big|_{V_\lambda}$ on diagonal

$\Rightarrow$ can invert this matrix and use the inverse matrix $D = C^{-1}$ to write a_λ in terms of $J_\mu(K)$.

__Warning__: $c_{\lambda\mu} \in \mathbb{Z}[q, q^{-1}]$ but the entries of $D = (d_{\lambda\mu})$ are in general rational functions in q!

When we combine RT invariants $J_\mu(K)$ with $d_{\lambda\mu}$, the result is $a_\lambda \in \mathbb{Z}[q, q^{-1}]$

$\Rightarrow$ deep divisibility results for RT invariants and their linear combinations

$$J_K(V_0, q) = 1 = J_K(V(1,1), q),\ \ J_K(V_1, q) = J_K(V(2,1), q) = 1 + q^2 + q^{-2} - q - q^{-1},$$

$$J_K(V_2, q) = 1 + q^3 + q^{-3} - q - q^{-1} + (q^3 + q^{-3} - q - q^{-1})(q^3 + q^{-3} - q^2 - q^{-2}).$$

$N = 2$

$K =$ figure 8 knot.

$$a_{2,1}(K) = -\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})} J_K(V_0, q) + \frac{q^{-3}}{(1-q^{-1})^3} J_K(V_1, q) -$$
$$\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-2})} J_K(V_2, q) - \frac{q^{-3}}{(1-q^{-1})^2(1-q^{-2})} J_K(V(1,1), q) +$$
$$\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})} J_K(V(2,1), q) = q^{-6}(-q^8 - q^7 - q^6 - q^5 + q^4 + 2q^3 + q^2 + q + 1).$$

$$\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})} J_K(V(2,1),q) = q^{-6}(-q^8-q^7-q^6-q^5+q^4+2q^3+q^2+q+1).$$

O knot.

coefs = $d_{\lambda\mu}$

These results generalize Habiro for sl_2.
(but get different expansion for gl_2)

Work much better for gl_N as opposed to sl_N

Can compute $c_{\lambda\mu}$, $d_{\lambda\mu}$ very explicitly.

Thm 2 Suppose that q is a root of unity. Then for all but finitely many λ and all μ we have $F_\lambda|_{V_\mu} = 0$. ($\Leftrightarrow F_\lambda|_{V_\mu}$ is divisible by $(1-q)\ldots(1-q^d)$ for some d depending on λ)

Cor We can apply this to study invariants of 3-manifolds
$M = S^3_1(K)$
Dehn surgery.

Then Witten–Reshetikhin–Turaev inv. of M is

$$\sum_\mu q^{\cdots} J_\mu(K) = \sum_\mu q^{\cdots} \sum_\lambda a_\lambda F_\lambda|_{V_\mu} = \text{(change order of summation)}$$

$$= \sum_\lambda a_\lambda \underbrace{\sum_\mu q^{\cdots} F_\lambda|_{V_\mu}}_{\text{if } q \text{ is a root of unity, vanishes for all but finitely many } \lambda.}$$

in $\mathbb{Z}[q,q^{-1}]$
$\Rightarrow$ can specialize to
[illegible]

specialize to root of 1.   many λ.

<u>Cor</u> WRT invariant of M is defined in Habiro ring $\widehat{\mathbb{Z}[q]} = \lim\limits_{n\to\infty} \frac{\mathbb{Z}[q]}{((1-q)(1-q^2)\cdots(1-q^n))}$

Proved earlier by Habiro for sl_2, Habiro-Le in general but by different methods.

<u>Idea of proof:</u> Rephrase the problem using symmetric functions: need symmetric polynomials $F_\lambda \in \mathbb{C}[x_1 \ldots x_N]^{S_N}$

$$F_\lambda(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}) = 0$$

for all μ not containing λ

$\Rightarrow$ interpolation Macdonald polynomials ($q=t$)

Okounkov, Olshanski, Knop, Sahi, ...

N=1 $\quad f_k(q^{-i}) = 0$ for $i < k$

$f_k(x) = (1-x)(1-qx)\cdots(1-q^{k-1}x)$ } Newton interp.

$$F_\lambda = \frac{\det(f_{\lambda_i+N-i}(x_j))}{\prod_{i<j}(x_i - x_j)} = q^{\cdot\cdot} s_\lambda + \text{lower terms}.$$

$C = (c_{\lambda\mu}) = F_\lambda(q^{-\mu_i-N+i})$ = triangular matrix

$(c_{\lambda\mu})$ $\quad {}_\lambda(q \quad)$ = triangular matrix

<u>Interpolation problem</u>: Given a values $F(q^{-\mu_i-N+i})$ for all μ, compute the function $F = \sum a_\lambda F_\lambda$.

$$F(q^{-\mu_i-N+i}) = \sum a_\lambda \underbrace{F_\lambda(q^{-\mu_i-N+i})}_{c}$$

$$a_\lambda = \sum d_{\lambda\mu} F(q^{-\mu_i-N+i})$$

$$D = (d_{\lambda\mu}) = C^{-1}$$

<u>Note</u> It is possible to write $d_{\lambda\mu}$ in terms of $c_{\lambda\mu}$ directly.

<u>Lemma</u> (Habiro) If $f(q) \in \widehat{\mathbb{Z}[q]}$ ← Habiro ring and $f(q)\cdot$(product of cyclotomic poly) $\in \mathbb{Z}[q,q^{-1}]$

Then $f(q) \in \mathbb{Z}[q,q^{-1}]$.

This is used to prove $a_\lambda \in \mathbb{Z}[q,q^{-1}]$.

<u>Q</u>: What is the meaning of interpolation Macdonald polynomials for general q,t?

No determinantal formulas, but lots of nice properties, $\quad F_\lambda = q^{\cdots} t^{\cdots} P_\lambda + \cdots$

Macdonald poly

Maybe related to categorification, t = homological grading?

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