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1. Results
\( K = \text{knot} \Rightarrow \overline{\text{HHH}(K)} = \text{reduced HOMFLYPT}\)
finite dim., triply graded (Khovanov-Rozansky) homology

Thm. \( \overline{\text{HHH}(K)} \) is symmetric:
\( \overline{\text{HHH}}_{i,-z,j,k}(K) \cong \overline{\text{HHH}}_{-i,-z,j,k+z,j} \)

Conjectured by Dunfield, Gukov, Rasmussen 2005

Other approaches: Oblomkov-Rozansky (HF in \( \text{Hilb}^n \mathbb{C}^2 \))
Galashin-Law (graded Koszul duality)

What about links ( unreduced homology?)
\( L \rightarrow \text{link with } c \text{ components} \Rightarrow \)
\( \text{HHH}(L) \) is a module over \( \mathbb{C}[x_1, \ldots x_c] \)
\( \text{HHH}(L) \) is a module over \( \mathbb{C}[x_1, \ldots, x_n] \)

**Problem:** Symmetry would break the action of \( x_i \).

**Solution:** "yification" (6.-Hogancamp)

\[ \text{HY}(L) = \text{"yified" link homology, module over } \mathbb{C}[x_1, \ldots, x_n] \]

**Thm 2** There exists a family of operators \( F_k, k \geq 1 \) in \( \text{HY}(L) \) such that:

- \( [F_k, F_k] = 0 \)
- \( [F_k, x_i] = 0, \quad [F_k, y_i] = i x_i^{k-1} \)
- \( F_2 \) satisfies "hard lefschetz condition" \( \odot \) and lifts to an action of \( \mathfrak{sl}(2) \)

**Cor** \( \text{HY}(L) \) is symmetric for any link,

- Symmetry exchanges \( x_i \leftrightarrow y_i \): \( [F_2, y_i] = 2x_i \)

**Cor** For knots, \( \text{HY}(K) = \overline{\text{HHH}(K)} \otimes \mathbb{C}[x, y] \)

\[ F_k = k x_i^{k-1} \frac{\partial}{\partial y} + F_k \]

for some operator \( F_k \) in \( \text{HHH} \)

- \( F_2 \) lifts to an action of \( \mathfrak{sl}(2) \) on \( \overline{\text{HHH}(K)} \) \( \Rightarrow \) symmetry.

**Ex:** (6.-Hogancamp) \( \overline{\text{HY}(T(u, v))} = \overline{\mathbb{C}(u, v) \text{ torus link}} \)

\[ = \bigwedge (x_i - x_i, y_i - y_i, \Theta_i - \Theta_i) \mathbb{C}[x_i - x_i, y_i - y_i] \]
\[ \bigwedge_{i \neq j} (x_i - x_j, y_i - y_j, \Theta_i - \Theta_j) \subset \mathbb{C}[x_1, x_2, y_1, y_2, \Theta_1, \Theta_2] \]

\[ F_k = \sum_{z_1} k^s \frac{\partial}{\partial y^s}, \text{ symmetry } x \leftrightarrow y \text{ is clear} \]

(1) Hard Left: \[ \deg F_2 = (0, 1, 2) \]

\[ F^+_2 : \mathcal{H}_i \mathcal{V}_j(L) \xrightarrow{\sim} \mathcal{H}_{i, z_1, k} \]

is an isomorphism.

(2) Idea of construction:

\[ \beta = \text{br} = \text{Rouquier construct} \]

\[ \text{a complex } A \text{- Soergel bimodule} \]

\[ (R-R \text{ bimodule}, R = \mathbb{C}[x_1, \ldots, x_n]) \]

such that:

(a) For any symmetric function \( f \)

\[ f(x_1, \ldots, x_n) = f(x_1', \ldots, x_n') \]

(b) Actions of \( x_i \) and \( x_i' \) are homotopic

\[ x_i - x_i' = [d, \xi_i] \]

\( w \) - permutation for \( \beta \)

\( 3_i = \text{chain homotopy} \)

Note: One can think of \( x_i, x_i' \)

as action of \( H^0(\text{unrav}) \) on braid

[211] \( \xrightarrow{\text{connect}} \) [2]
More abstractly: \( B = R \otimes R = \mathbb{C} \left[ x_1 - x_n^* \right] \to R^g \), \( f(x) = f(x^*) \) for symmetric \( f \).

Can write \( x_1^r + \ldots + x_n^r = (x_1^r)^r + \ldots + (x_n^r)^r \) for symmetric \( f \).

\[ A = \text{resolution of } R \text{ over } B : \]

\[ d(\xi_i) = x_i - x_i' \]

\[ B \langle \xi_i \rangle \to B \to R = \mathbb{C} \left[ x_1^r \ldots x_n^r, x_i - x_i' \right] \]

\[ (x_i = x_i') \]

Observe: \( d \left( \sum \xi_i (x_i + x_i') \right) = \sum (x_i - x_i') (x_i + x_i') = \]

\[ = \sum (x_i^2 - (x_i')^2) = 0 \]

\( \Rightarrow \) use \( d \) as an element \( \omega_2 : d(\omega_2) = \sum \xi_i (x_i + x_i') \)

More generally, use \( \omega_k : d(\omega_k) = \sum \xi_i (x_i^{k-1} + \ldots + x_i) \)

\[ x_i^{k-1} = \frac{(x_i - x_i')^k}{x_i - x_i'} \]

Thus, \( A = \text{free commutative dga generated over } B \)

by \( \xi_i, \omega_k : d(\xi_i) = x_i - x_i' \)

\[ d(\omega_k) = \sum \xi_i \cdot \omega_{k-1}(x_i, x_i') \]

Thus \( A \cong A \otimes A \) homotopy equivalence

\( \Rightarrow \) coproduct \( \Delta : A \to A \otimes A \)

coassociative up to homotopy.
Thus, for any braid $\beta$, the corresponding Rouquier complex $T_\beta$ has an action of $A$:

- $x_i$: act as above
- $x_i$: are twisted by $w \mapsto x_i w_i$
- $z_i$: "dot-string homotopies"
- $u_k$: some assembly operators.

Proof: Can check for $\chi - \chi$, $u_k = 0$ for all $k$.

We reproduce to extend to arbitrary braids $\beta$.

**y-ification:** $T_\beta \sim T_\beta \otimes \mathbb{C} \{ y_1, -y_n \}$

$$D = d + \sum \xi_i y_i$$

$$HY = H^0( HH^+( T_\beta \otimes \mathbb{C} \{ y_1, -y_n \} ))$$

$$F_K = \sum \frac{1}{k-1} (x_i, x_i) \frac{\partial}{\partial y_i} + u_k$$

Exercise: $[D, F_K] = 0$! $[d, u_k] = 0$, equally.

Checking commutation relations: straightforward.

We are well defined up to homotopy $\Rightarrow$
we are well defined up to homotopy \(\Rightarrow\)
\[ \Rightarrow F_x \text{ well defined up to homotopy.} \]

**Hard Leftshtz:** \( A^\circ \) is bigraded complex \( \otimes \) F chain map of degree \((q,2)\)

\[ d = (0,1) \]

Say that \( F \) is leftshtz if \( F^j : H_z(A) \to H_z(A) \)

\[ F(Q) \cdot F(Q) = Q \text{ is an iso.} \]

**Lemma:** If \( 0 \to A \to B \to C \to 0 \)

exact triple commute with \( F \)

\( F \) is leftshtz for \( A \) and \( C \) \( \Rightarrow \) leftshtz for \( B \).

**Proof:** \( 5 \text{-lemma!} \)

Use seair relations (seair exact triangles)

+ Markov towers

+ to reduce \( L \) to unlinics.

\( \bigotimes (HY(\text{unlinic}) \otimes \text{exterior algebra} ) \)

Check for such "building blocks", use lemma to conclude in general.

**Q:** Is it possible to construct \( E \) on chain level.

\[ V = \text{vector space w. symm. form} \ W \]

\[ \wedge V \otimes W \text{ leftshtz.} \]