Tautological classes and symmetry in HOMFLY homology

arXiv:2103.01212

Tuesday, April 6, 2021 12:27 PM

From <https://arxiv.org/abs/2103.01212>

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① **Results** K = knot $\Rightarrow$ $\overline{HHH}(K)$ = reduced HOMFLYPT (Khovanov-Rozansky) homology

finite dim., triply graded

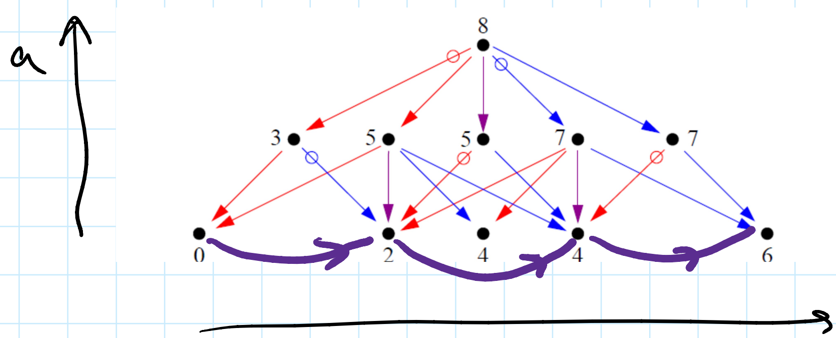
Thm $\overline{HHH}(K)$ is symmetric:

$$\overline{HHH}_{i,-2j,k}(K) \xrightarrow[\sim]{F^j} \overline{HHH}_{i,2j,k+2j}$$

a-grading, q-grading, t-grading.

$\deg F = (0,4,2)$

Conjectured by Dunfield, Gukov, Rasmussen ~ 2005



T(3,4)

Other approaches: Oblomkov-Rozansky (MF on $Hilb^n \mathbb{C}^2$)

Galashin-Lam (graded Koszul duality)

What about links / unreduced homology?

$L \rightarrow$ link with c components $\Rightarrow$

$HHH(L)$ is a module over $\mathbb{C}[x_1 \ldots x_c]$

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Problem: Symmetry would break the action of x_i

Solution: "y-ification" (G.-Hogancamp)

$HY(L)$ = "y-ified" link homology,

module over $\mathbb{C}[x_1 \dots x_c, y_1 \dots y_c]$

Thm 2 There exists a family of operators F_k, $k \geq 1$ in $HY(L)$ such that:

- $[F_k, F_\ell] = 0$
- $[F_k, x_i] = 0$, $[F_k, y_i] = k x_i^{k-1}$
- F_2 satisfies "hard Lefshetz condition"

(*) and lifts to an action of $\mathfrak{sl}(2)$, $F_2 = F$

Cor $HY(L)$ is symmetric for any link,

Symmetry exchanges $x_i \leftrightarrow y_i$ ($[F_2, y_i] = 2x_i$)

Cor For knots, $HY(K) = \overline{HHH}(K) \otimes \mathbb{C}[x, y]$

$$F_k = k x^{k-1} \frac{\partial}{\partial y} + \overline{F}_k$$

for some operator $\overline{F}_k$ on $\overline{HHH}$

$\overline{F}_2$ lifts to an action of $\mathfrak{sl}(2)$ on $\overline{HHH}(K)$

$\Rightarrow$ symmetry.

Ex: (G.-Hogancamp) $HY(T(n, n)) =$

(n, n) torus link

$= \bigcap (x_i - x_j, y_i - y_j, \theta_i - \theta_j) \subset \mathbb{C}[\overbrace{x_1 \dots x_n, y_1 \dots y_n,}^{\text{even}}$

$$= \bigcap_{i \neq j} (x_i - x_j, y_i - y_j, \theta_i - \theta_j) \subset \mathbb{C}[\overbrace{x_1 \dots x_n, y_1 \dots y_n}^{\text{even}}, \underbrace{\theta_1 \dots \theta_n}_{\text{odd}}]$$

ideal $\xrightarrow{\sim}$ HY(unlink)

$F_k = \sum_{i=1}^{n} k x_i^{k-1} \frac{\partial}{\partial y_i}$, symmetry $x \leftrightarrow y$ is clear

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(*) Hard Lefshetz: $\deg F_2 = (0, 4, 2)$

$$F_2^j : HY_{i,-2j,k}(L) \xrightarrow{\sim} HY_{i,2j,k+2j}(L)$$

is an isomorphism.

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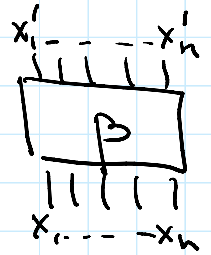
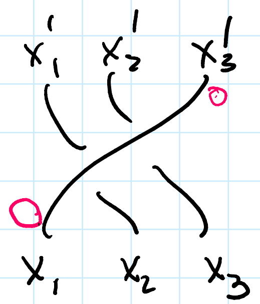
② Idea of construction:



β = braid $\rightsquigarrow$ Rouquier constructs a complex T_β of Soergel bimodules (R-R bimodules, $R = \mathbb{C}[x_1 \dots x_n]$)

such that (a) For any symmetric function f

$$f(x_1 \dots x_n) = f(x_1' \dots x_n')$$



(b) Actions of x_i and x_i' are homotopic $x_i - x'_{w(i)} = [d, \xi_i]$

w = permutation for β

ξ_i = chain homotopy

$x_1 \sim x_3'$

$x_2 \sim x_1'$

$x_3 \sim x_2'$

__Note:__ One can think of x_i, x_i' as action of H^*(unknot) on braid

$R \otimes \bigcirc \xrightarrow{\text{connect}} R$

$$\mathcal{B} \sqcup \bigcirc \xrightarrow[\text{sum}]{\text{connect}} \mathcal{B}$$

<u>More abstractly</u>: $\mathcal{B} = R \otimes_{R^{S_n}} R = \dfrac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{(f(x) = f(x'))}$ for symmetric f

Can write $x_1^k + \dots + x_n^k = (x_1')^k + \dots (x_n')^k$

$\mathcal{A}$ = resolution of R over $\mathcal{B}$:

$$d(\xi_i) = x_i - x_i'$$

$$\begin{matrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{matrix} \qquad \mathcal{B}\langle \xi_i \rangle \longrightarrow \mathcal{B} \longrightarrow R = \frac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{(x_i = x_i')}$$

Observe: $d\left(\sum \xi_i (x_i + x_i')\right) = \sum (x_i - x_i')(x_i + x_i') =$

$$= \sum \left(x_i^2 - (x_i')^2\right) = 0$$

$\Rightarrow$ need an element u_2: $d(u_2) = \sum \xi_i (x_i + x_i')$

More generally, need u_k: $d(u_k) = \sum \xi_i \left(x_i^{k-1} + \dots + (x_i')^{k-1}\right)$

$$\frac{x_i^k - (x_i')^k}{x_i - x_i'} = h_{k-1}(x_i, x_i')$$

<u>Thm</u> $\mathcal{A}$ = free commutative dga generated over $\mathcal{B}$ by ξ_i, u_k: $d(\xi_i) = x_i - x_i'$

$$d(u_k) = \sum \xi_i \, h_{k-1}(x_i, x_i').$$

<u>Thm</u> $\mathcal{A} \simeq \mathcal{A} \otimes_R \mathcal{A}$ homotopy equivalence

$\Rightarrow$ coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes_R \mathcal{A}$

Coassociative up to homotopy

<u>Note</u>: explicit homotopy inverse
$\varepsilon : \mathcal{A} \to R$
$\mathcal{A} \otimes \mathcal{A} \xrightarrow{\varepsilon \otimes 1} \mathcal{A}$
$\varepsilon \otimes 1 \simeq 1 \otimes \varepsilon$

coassociative up to homotopy $\varepsilon\otimes 1 \simeq 1\otimes\varepsilon$

Thm For any braid β, the corresponding Rouquier complex T_β has an action of $\mathcal{A}$:

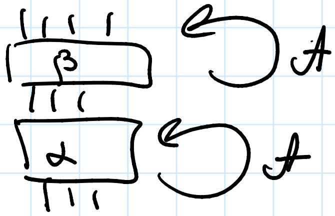
x_i act as above

x'_i are twisted by $w \longleftrightarrow x'_{w(i)}$

$\xi_i \longleftrightarrow$ "dot-sliding homotopies"

$u_k \longrightarrow$ some interesting operators.

Proof: Can check for the two crossings, $u_k = 0$ for all k.



$\mathcal{A}\otimes\mathcal{A}$ act on $\alpha\otimes\beta$

$\mathcal{A} \xrightarrow{\Delta} \mathcal{A}\otimes\mathcal{A}$

use coproduct to extend to arbitrary braids ∎.

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y-ification: $T_\beta \rightsquigarrow T_\beta\otimes\mathbb{C}[y_1\ldots y_n]$ ← formal variables

$$D = d + \underline{\sum \xi_i y_i}$$

$$HY = H^*(HH^*(T_\beta\otimes\mathbb{C}[y_1\ldots y_n]), D)$$

$$F_k = \sum \underline{h_{k-1}(x_i, x'_i)}\frac{\partial}{\partial y_i} + u_k$$

Exercise: $[D, F_k] = 0$! $[d, u_k]\neq 0$, cancels

$\xi_i h_{k-1}(\cdot\cdot)[y_i, \frac{\partial}{\partial y_i}]$

Checking commutation relations: straightforward

u_k are well defined up to homotopy $\Rightarrow$

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$\Rightarrow F_k$ well defined up to homotopy.

<u>Hard Lefshetz</u>: A° = bigraded complex $d=(0,1)$ $\circlearrowleft F$ chain map of degree $(4,2)$

Say that F is lefshetz if $F^j: H_{-2j,k}(A) \to H_{2j,k+j}(A)$ is an iso.

<u>Lemma</u>: If $0 \to \overset{F\circlearrowleft}{A^\bullet} \to \overset{F\circlearrowleft}{B^\bullet} \to \overset{F\circlearrowleft}{C^\bullet} \to 0$ exact triple commute with F

F is lefshetz for A and C $\Rightarrow$ lefshetz for B.

<u>Proof</u>: 5-lemma!

Use skein relations (skein exact triangles)

+ Markov moves

to reduce L to unlinks.

($\leftarrow HY(\text{unlink}) \otimes (\text{exterior algebra})$)

Check for such "building blocks", use lemma to conclude in general!

<u>Q</u>: Is it possible to construct E on chain level.

V = vector space w. symp. form ω

$\Lambda^\bullet V \circlearrowleft \wedge\omega$ lefshetz.