Plan

(1) Theorems and Conjectures
(2) Main Conjecture
(3) Intro to knot invariants

[redacted]

(1) Conj 1 (G., 2010) [2] The Poincaré polynomial for ($a=0$) part of the Khovanov–Rozansky homology[⊛] of the $(n, n+1)$ torus knot $=$ q,t Catalan number $c_n(q,t)$

$$c_n(q,t) = (\nabla e_n, e_n) = P_{T(n,n+1)}(q,t,0)$$ ⊛ notations for KhR

[5] Thm (Hogancamp, 2017) Conj. is true! (see next talk!)

[3] Conj 2 (G., Negut '13) For m and n coprime,

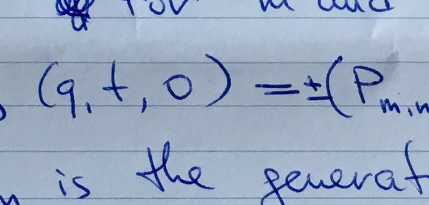
$$P_{T(m,n)}(q,t,0) = \pm(P_{m,n} \cdot 1, e_n)$$

where $P_{m,n}$ is the generator of the Elliptic Hall algebra defined yesterday.

[3] Th (G., Negut): RHS = "refined Chern-Simons invariant" defined by Aganagic-Shakirov + Cherednik.

[6] Th (Mellit, '17) (a) Conj. 2 is true.

(b) RHS = sum over Dyck paths in $m \times n$ rectangle



$$(P_{m,n} \cdot 1, e_n) = \sum_D q^{\text{area}} t^{\text{dinv}}$$

(c) $\pm P_{m,n} \cdot 1$ is Schur positive, and has an explicit combinatorial description
$\Rightarrow$ "rational shuffle conjecture" is true.

All these are special cases of a bigger picture:

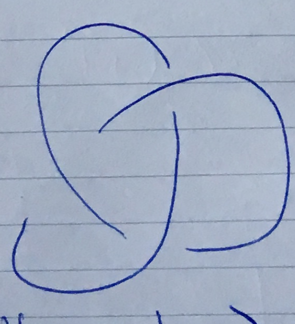
[4] Conj (G., Neguț, Rasmussen'16) For every link L there is a vector bundle (or sheaf, or complex of sheaves) $\mathcal{F}_L$ on the Hilbert scheme of points in $\mathbb{C}^2$ such that $H^*(\mathrm{Hilb}^n\mathbb{C}^2, \mathcal{F}_L) \simeq KhR(L)$

We have 2 potential constructions of $\mathcal{F}_L$, see GNR, and Oblomkov–Rozansky, conjecturally the same.

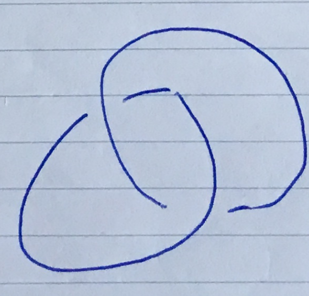
By the work of Haiman, $\mathcal{F}_L \rightsquigarrow$ some symmetric function $\in \Lambda(q,t)$.

For the rest of the talk, I will focus on this $\boxed{\text{Ex: } P_{m,n} \cdot 1}$ and explain some basic definitions for link invariants, and why symmetric functions appear in the picture.

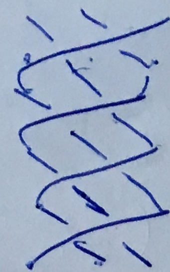
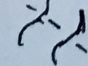
Knot



link

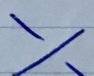
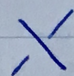


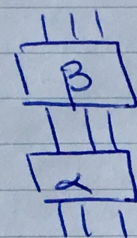
Thm (Alexander) Every link is a closure of a braid.

 braid 

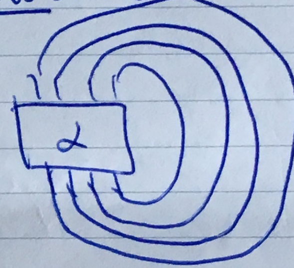
$(\sigma_2\sigma_1)^4$

Braid group: $\langle \sigma_1 \ldots \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
$\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2 \rangle$
$\shortparallel$
Br_n

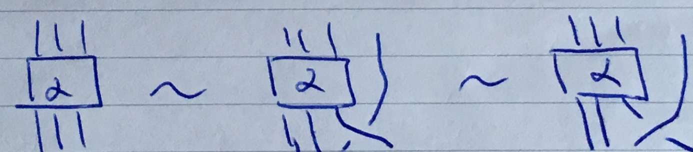
 σ_i  σ_i^{-1}

Composition  $\beta\alpha \neq \alpha\cdot\beta$

Closure:



Thm (Markov) Two braids have the same closure iff they are related by a sequence of moves:

(M2) 

(M1) $\alpha\beta \sim \beta\alpha$.

Corollary: A link invariant = a braid invariant
(= function $Br_n \longrightarrow ?$)
which is constant on equivalence classes for (M1, M2).

How to construct such invariants?

(M1) is easy: pick a representation of Br_n
V, $\alpha \longrightarrow Tr_V \alpha$., $Tr_V(\alpha\beta) = Tr_V(\beta\alpha)$.

<u>Concretely</u>: $H_n = \langle T_1 \ldots T_n \mid (T_i - 1)(T_i + q) = 0$
$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
$T_i T_j = T_j T_i \quad |i-j| \geq 2 \rangle$

H_n is a q-deformation of $\mathbb{C}[S_n]$
and has the same rep. theory for generic q.

$Br_n \longrightarrow H_n$
$\sigma_i \longrightarrow T_i$

$|\lambda| = n \Rightarrow V_\lambda$ = irrep of H_n.

$$\beta \longrightarrow \beta \longrightarrow \sum_\lambda Tr_{V_\lambda}(\beta) \cdot S_\lambda = f_\beta \in \Lambda(q)$$

Schur functions

symmetric function deg $= n$

f_β = "universal trace" of β, satisfies (M1)

[1] <u>Th</u> (Jones, Ocneanu) For all N, $\beta \longrightarrow f_\beta(1, q, \ldots q^{N-1})$ also satisfies (M2)

$$= f_\beta\left[p_k = \frac{1-q^N}{1-q} = \frac{1-a}{1-q}\right]$$

$a = q^N$

<u>Rmk</u> Sometimes it is easier to work with $\tilde{f}_\beta = \omega f_\beta \left[\frac{X}{1-q}\right]$.

CENTRE
DE RECHERCHES
MATHÉMATIQUES

Example $T(m,n) = (\sigma_1 \cdots \sigma_{n-1})^m$

Thm (Jones, Rosso) $f_{T(m,n)} = s_{(n)} - q^m s_{(n-1,1)} + q^{2m} s_{(n-2,1,1)} - \cdots \pm q^{m(n-1)} s_{(1^n)}$

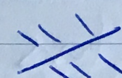
Thm (G., Oblomkov, Rasmussen, Shende)

$$\tilde{f}_{T(m,n)} = \text{Frobenius character of } L_{\frac{m}{n}}$$

$L_{\frac{m}{n}}$: irred. rep. of Rational Cherednik algebra

$\Rightarrow$ Schur positive!

Example $\tilde{f}_{T(1,n)} = \dfrac{e_n}{1-q}$



$FT_n = (\sigma_1 \cdots \sigma_{n-1})^n$ = full twist

central in $Br_n \longrightarrow$ central in H_n

Fact FT_n acts in V_λ by a scalar $q^{\sum_\lambda c(\square)}$, where $\sum_\lambda c(\square)$ is the sum of contents of boxes in λ.

So: $f_{\beta \cdot FT_n} = \sum_\lambda Tr(\beta \cdot FT_n)|_{V_\lambda} \cdot s_\lambda = \sum_\lambda q^{\sum_\lambda c(\square)} Tr(\beta)|_{V_\lambda} \cdot s_\lambda$

$\Rightarrow \tilde{f}_{\beta \cdot FT_n} = \nabla_{q=1/t} \tilde{f}_\beta$, where $\nabla_{q=1/t}$ is diagonal in basis $s_\lambda\left[\frac{X}{1-q}\right] = \tilde{H}_\lambda(q, q^{-1})$ with eigenvalues $q^{\sum_\lambda c(\square)}$

Corollary $\tilde{f}_{T(n,n+1)} = \nabla_{q=1/t} \tilde{f}_{T(1,n)} = \nabla_{q=1/t} e_n$.

www.crm.math.ca

References:

(1) V. Jones. Hecke algebra representations of braid groups and link polynomials. Ann. of Math. 126.2 (1987), 335-388.

(2) E. Gorsky. q,t-Catalan numbers and knot homology. Zeta functions in algebra in geometry 213-232, Contemp. math. 566. arXiv: 1003.0916

(3) E. Gorsky, A. Neguț. Refined knot invariants and Hilbert schemes. Journal de Mathématiques pures et appliquées 104 (2015), 403-435. arXiv: 1304.3328

(4) E. Gorsky, A. Neguț, J. Rasmussen. Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology. arXiv: 1608.07308

(5) M. Hogancamp. Khovanov-Rozansky homology and higher Catalan sequences. arXiv: 1704.01562.

(6) A. Mellit. Homology of torus knots. arXiv: 1704.07630

(7) E. Gorsky, A. Oblomkov, J. Rasmussen, V. Shende. Torus knots and the rational DAHA. Duke Math J. 163, no. 14 (2014) 2709-2794. arXiv: 1207.4523

(8) A. Oblomkov, L. Rozansky. Knot homology and sheaves on the Hilbert scheme of points on the plane. arXiv: 1608.03227