Plan
① Theorems and Conjectures
② Main Conjecture
③ Intro to knot invariants

① Conj.1 [G., 2010]:
The Poincaré polynomial for (a = 0) part of the Khovanov–Rozansky homology of the \((n, n+1)\) torus knot \(q+n\) Catalan number \(C_n(q, t)\)

\[
C_n(q, t) = \langle \Delta_{n}, e_n \rangle = P_{T(a, m+1)}(q, t, 0)
\]

[5] Thm. (Hosea, 2017) Conj. is true! (see next talk!)

[3] Conj.2 [G., Negut, ’13]:
For \(m = n + 1\) coprime,

\[
P_{T(m,n)}(q, t, 0) = P_{m} \cdot 1, e_n
\]

where \(P_{m} \cdot n\) is the generator of the Elliptic Hall algebra defined yesterday.


defined by Aganagic–Shadursov–Cherednik.


(6) RHS = sum over Dyck paths in \(m \times n\) rectangle

\[
(P_{m,n} \cdot 1, e_n) = \sum_{a} q^{area} + \sum_{b} \sum_{c} q^{area}
\]

(c)\(P_{m,n} \cdot 1\) is Schur positive, and has an explicit combinatorial description

⇒ "Rational Shuffle Conjecture" is true.
All these are special cases of a bigger picture:

**Conj.** (G. Negut, Rasmussen) For every link \( L \), there is a vector bundle (or sheaf, or complex of sheaves) \( F_L \) on the Hilbert scheme of points in \( \mathbb{C}^2 \), such that \( \mathbb{H}^*(\text{Hilb}^* \mathbb{C}^2, F_L) \cong \text{Kh} R(L) \)

We have 2 potential constructions of \( F_L \).

- GNR, and Oblomov - Rozansky, conjecturally the same.

By the work of Haiman, \( F_L \) is some symmetric function \( \Delta(g(t)) \).

For the rest of the talk, I will focus on this and explain some basic definitions for link invariants, and why symmetric functions appear in the picture.

**Thm.** (Alexander) Every link is a closure of a braid.
Braid group: \( < \sigma_1, \ldots, \sigma_{n-1} : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \neq 2 > \)

\( \times \sigma_i \times \sigma_i^{-1} \)

Composition : \( B \times \beta \neq \beta \times B \)

Thm (Markov) Two braids have the same closure iff they are related by a sequence of moves:

(M2) \( \begin{array}{c} 1 \\ 2 \end{array} \sim \begin{array}{c} 2 \\ 1 \end{array} \sim \begin{array}{c} 2 \\ 1 \end{array} \)

(M1) \( \alpha \beta \sim \beta \alpha \)

Corollary: A link invariant = a braid invariant (= function \( Br_n \rightarrow ? \)) which is constant on equivalence classes for \((M1, M2)\).

How to construct such invariants?
(M1) is easy: pick a representation of $Br_{n+N}$
\[ V \to V \to Tr_{V} V, \quad Tr_{V}(x + p) = Tr_{V}(x + 2p). \]

Concretely: $H_{n} = \langle T_{i} \mid (T_{i}^{*} - 1)(T_{i} + q) = 0 \rangle$
\[ T_{i} T_{j} = T_{j} T_{i} \text{ if } |i - j| \geq 2. \]

$H_{n}$ is a $q$-deformation of $C(B)$
and has the same rep. theory for generic $q$.

$Br_{n} \to H_{n} \to V_{x} \to \text{Rep of } H_{n}.$

$\beta \to \sum_{x} \text{Tr}_{V_{x}}(\beta) \cdot s_{x} = \sum_{\beta \in \Delta(q)} \text{Schur function}$

$fb = \text{"universal trace" of } \beta, \text{ satisfies (M1)}$

[I] Th (Jones, Ocneanu) For all $N$, $\beta \to fb(1, q, \ldots, q^{N-1})$
also satisfies (M2)

$fb \left[ p_{i} = \frac{1 - q^{N}}{1 - q} \right] = \frac{1 - q^{N}}{1 - q} \cdot p_{i}$

Rmk: Sometimes it is easier

do work with $\tilde{fb} = w_{fb} \left[ \frac{x}{1 - q} \right].$
Example $T(n, m) = (\sigma_1 \cdots \sigma_m)^m$

Thin (Jones, Rosso) $f_T(n, m) = S_m^{\mathbb{Q}_{m+1}} - q^m S_m^{\mathbb{Q}_{m-1}} + q^m S_m^{\mathbb{Q}_{m-2}} - \cdots + q^m S_m^{\mathbb{Q}_{m-k}}$

Thin (G. O. Bloomenthal, Ram, V. S. Sheeple)

$\hat{f}_{T(n, m)}$ = Frobenius character of $L_m$

irred. rep. of Rational Chevalley algebra

$\Rightarrow$ Schur positive!

Example $\hat{f}_{T(1, m)} = e_m \frac{1}{1-q}$

$\hat{F}_{T(n)} = (\sigma_1 \cdots \sigma_n)^n$ = full twist

Central in $B_{\mathfrak{u}} \rightarrow$ central in $\mathfrak{u}$

Fact $F_{T(n)}$ acts in $V_\lambda$ by a scalar $q^{\sum_{C(\lambda)}}$

where $\sum_{C(\lambda)}$ is the sum of content of boxes in $\lambda$.

So: $\sum_{\lambda} f_{\beta} \cdot F_{T(n)} = \sum_{\lambda} \text{Tr}(\beta \cdot F_{T(n)})|_{V_\lambda} \cdot S_{\lambda} = \sum_{\lambda} q^{\sum_{C(\lambda)}} \text{Tr}(\beta)|_{V_\lambda} \cdot S_{\lambda}$

$\Rightarrow \hat{f}_{\beta} \cdot F_{T(n)} = \nabla_{q^{1/4}} \hat{f}_{\beta}$, where $\nabla_{q^{1/4}}$ is diagonal

in basis $S_{\lambda} \left[ \frac{x}{1-q} \right] = \hat{f}_{\lambda}(q, q^{-1})$ with eigenvalues $L_q^{\sum_{C(\lambda)}}$

Corollary $\hat{f}_{T(n, m+1)} = \nabla_{q^{1/4}} \hat{f}_{T(n, m)} = \nabla_{q^{1/4}} e_n$. 

[CRM De Recherches Mathematiques]
References:


