Motivic Poincare series and knot homology

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Poincare series and zeta functions

$C = \cup_{i=1}^{r} C_i$ - plane curve singularity at the origin in $\mathbb{C}^2$.
$\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$ - uniformisations of its components.

$$v_i(f) = \mathrm{Ord}_0(f(\gamma_i(t)))$$

One can define $\mathbb{Z}^r$-indexed filtration

$$J_{\underline{v}} = \{f \in \mathcal{O} | v_i(f) \geq v_i\}.$$

Consider the Laurent series

$$L_C(t_1, \ldots, t_r) = \sum_{\underline{v}} t_1^{v_1} \ldots t_r^{v_r} \cdot \dim J_{\underline{v}} / J_{\underline{v}+\underline{1}}.$$

The Poincare series of the curve C is defined by the formula

$$P_C(t_1, \ldots, t_r) = \frac{L_C(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r}(t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}$$

If $r = 1$, we get

$$P_C(t) = \sum_{v=0}^{\infty} t^v \cdot \dim J_v / J_{v+1}.$$

Proposition(Campillo, Delgado, Gusein-Zade)

$$P_C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r^{v_r} d\chi$$

$\Delta_C(t_1, \dots, t_r)$ - multi-variable Alexander polynomial of the link of C

Theorem (CDG)

If $r = 1$, then

$$P_C(t)(1-t) = \Delta_C(t),$$

and if $r > 1$, then

$$P_C(t_1, \dots, t_r) = \Delta_C(t_1, \dots, t_r).$$

Motivic Poincare series

In analogy to the construction of the motivic measure on the space of arcs, one can define a motivic measure on the ring $\mathcal{O}$ approximating it by jet spaces.
Motivic Poincare series is the motivic integral

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r^{v_r} d\mu$$

Let $q = \mathbb{L}^{-1}$ be a formal variable. Let $h(\underline{v}) = \mathrm{codim} J_{\underline{v}}$, and

$$L_g(t_1, \ldots, t_r, q) = \sum_{\underline{v} \in \mathbb{Z}^r} \frac{q^{h(\underline{v})} - q^{h(\underline{v}+\underline{1})}}{1-q} \cdot t_1^{v_1} \ldots t_r^{v_r}.$$

Theorem (CDG)

$$P_g^C(t_1, \ldots, t_r; q) = \frac{L_g^C(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}$$

If $r = 1$, we have

$$P_g^C(t) = \sum_{v=0}^{\infty} t^v \cdot \frac{q^{codim J_v} - q^{codim J_{v+1}}}{1-q}.$$

Irreducible case

One can prove that

$$P_C(t) = 1 + t^{\sigma_1} + t^{\sigma_2} + t^{\sigma_3} + \dots,$$

where $\{0, \sigma_1, \sigma_2, \dots\}$ form the semigroup of C. Then

$$P_g^C(t; q) = 1 + qt^{\sigma_1} + q^2 t^{\sigma_2} + q^3 t^{\sigma_3} + \dots.$$

Example. $C = \{x^3 = y^5\}$.

$$P_C(t) = 1 + t^3 + t^5 + t^6 + t^8 + t^9 + \dots = \frac{(1 - t^{15})}{(1 - t^3)(1 - t^5)},$$

Therefore

$$P_g^C(t; q) = 1 + qt^3 + q^2 t^5 + q^3 t^6 + q^4 t^8 + q^5 t^9 + \dots$$

Properties

The reduced motivic Poincare series is the power series

$$\overline{P}_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P_g(t_1, \ldots, t_r).$$

Theorem (-)

1. **Polynomiality.** $\overline{P}_g(t_1, \ldots, t_n; q)$ *is a polynomial in* $t_1, \ldots, t_n$ *and* q.
2. **Reduction to the Alexander polynomial.** *If* $n = 1$, *then*

$$\overline{P}_g(t; q = 1) = \Delta(t),$$

where Δ *denote the Alexander polynomial of the link of the corresponding plane curve singularity. If* $n > 1$, *then*

$$\overline{P}_g(t_1, \ldots, t_n; q = 1) = \Delta(t_1, \ldots, t_n) \cdot \prod_{i=1}^{n} (1 - t_i).$$

3. **Forgetting components.** Let C be a curve with n components, and C_1 be an irreducible curve. Then

$$\overline{P}_g^{C\cup C_1}(t_1,\ldots,t_n,t_{n+1}=1)=(1-q)\overline{P}_g^{C}(t_1,\ldots,t_n).$$

If C has only one component, then

$$\overline{P}_g^{C}(t=1)=1.$$

4. **Symmetry.** Let μ_α be the Milnor number of C_α, $(C_\alpha\circ C_\beta)$ is the intersection index of $C_\alpha\circ C_\beta$, $\mu(C)$ is the Milnor number of C. Let

$$l_\alpha=\mu_\alpha+\sum_{\beta\neq\alpha}(C_\alpha\circ C_\beta),\quad \delta(C)=(\mu(C)+r-1)/2.$$

Then

$$\overline{P}_g(\frac{1}{qt_1},\ldots,\frac{1}{qt_r})=q^{-\delta(C)}\prod_\alpha t_\alpha^{-l_\alpha}\cdot\overline{P}_g(t_1,\ldots,t_r).$$

General case: algorithm

For a proper everywhere set P we define $\widetilde{H}_P$ - explicitly given polynomial divisible by $\prod_{i\in E(P)}(1-u_i)$

Theorem

For a proper everywhere set P define the numbers $d_P(n)$ by the equation

$$H_P(u)=\sum_n d_P(n)u^n d_P(n)=$$

$$\frac{\prod_{i\in P}[(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}]}{\prod_{i\in E(P)}(1-u_i)}\widetilde{H}_P(u_1,\ldots,u_s).$$

Then

$$\overline{P}_g(t_1,\ldots,t_r)=\sum_{P\in\mathcal{P}}(-1)^{|P|}q^{|P|}t_P\times\sum_n d_P(n)t^{Mn}q^{F(n)-\sum n_i}.$$

General case: examples

Consider the singularity $x^{k_0} - y^{k_0} = 0$.
For $0 < k < k_0$ let the numbers $c_k(n)$ be defined by the equation

$$A_k(u) = \sum_{n=0}^{\infty} u^n c_k(n) = (1-uq)^{k_0-k-1}(1-u)^{k-1},$$

and for $k = 0$ let the numbers $c_0(n)$ be defined by the equation

$$A_0(u) = \sum_{n=0}^{\infty} u^n c_0(n) = \frac{(1-uq)^{k_0-1} - u(u-q)^{k_0-1}}{1-u}.$$

$$\overline{P}_g(t_1, \ldots, t_{k_0}) = \sum_{K \subset_{\neq} K_0} (-1)^{|K|} q^{|K|} t_K \sum_{n=0}^{\infty} c_{|K|}(n)(t_1 \ldots t_{k_0})^n q^{\frac{n(n+1)}{2}}.$$

For example, if $k_0 = 2$,

$$A_1(u) = 1, A_0(u) = \frac{1 - uq - u(u-q)}{1-u} = 1 + u,$$

so

$$\overline{P}_g(t_1, t_2) = 1 - qt_1 - qt_2 + qt_1 t_2.$$

If $k_0 = 3$,

$$A_1(u) = 1 - qu, A_2(u) = 1 - u, A_0(u) = 1 + (1 - 2q - q^2)u + u^2,$$

so

$$\overline{P}_g(t_1, t_2, t_3) = 1 - q(t_1 + t_2 + t_3) + q^2(t_1 t_2 + t_1 t_3 + t_2 t_3) +$$

$$q(1 - 2q - q^2)t_1 t_2 t_3 + q^3 t_1 t_2 t_3 (t_1 + t_2 + t_3) -$$

$$q^3 t_1 t_2 t_3 (t_1 t_2 + t_1 t_3 + t_2 t_3) + q^3 t_1^2 t_2^2 t_3^2.$$

Heegard-Floer homology

Heegard-Floer homology were introduced by P. Ozsvath and Z. Szabo. To each link $L = \cup_{i=1}^{r} K_i$ they assign the collection of homology groups $\widehat{HFL}_d(L, \underline{h})$, where d is an integer and $\underline{h}$ belongs to some r-dimensional lattice.
They give a "categorification" of the Alexander polynomial of L: if $r = 1$, then

$$\sum_h \chi(\widehat{HFL}_*(L, h))t^h = \Delta^s(t),$$

where $\Delta^s(t) = t^{-\deg \Delta/2}\Delta(t)$ is the symmetrized Alexander polynomial of L. If $r > 1$, then

$$\sum_{\underline{h}} \chi(\widehat{HFL}_*(L, h))\underline{t}^h = \prod_{i=1}^{r}(t_i^{1/2} - t_i^{-1/2}) \cdot \Delta^s(t_1, \dots, t_r).$$

Theorem (Ozsvath,Szabo)

Let $g(K)$ be the genus of a knot K, i.e. the minimal genus of a Seifert surface for K. Then

$$g(K) = \max\{n | \dim \widehat{HFL}_*(K, n) \neq 0\}$$

Theorem (Ni)

A knot K is fibered if and only if

$$\dim \widehat{HFL}_*(L, g(K)) = 1.$$

Consider the ring $R = \mathbb{Z}[U_1, \ldots, U_r]$. For every r-component link L there exists a $\mathbb{Z}^r$-filtered chain complex $CFL^-(S^3, L)$ of R-modules, whose filtered homotopy type is an invariant of the link L. The operators U_i lowers the homological grading by 2 and the filtration level by $\underline{1}$.

$$\widehat{CFL}(S^3, L) = CFL^-(S^3, L)/(U_1 = \ldots = U_r = 0)$$

$$H^*(CFL^-(S^3, L)) = H^*(CFL^-(S^3)) = \mathbb{Z}[U]$$

$$H^*(CFL^-(S^3, L, k)/CFL^-(S^3, L, k-1)) = HFL^-(S^3, L, k)$$

$$H^*(\widehat{CFL}(S^3, L, k)/\widehat{CFL}(S^3, L, k-1)) = \widehat{HFL}(S^3, L, k)$$

Let K be the link of an irreducible curve singularity C. Consider the Poincare polynomial for the Heegard-Floer homologies:

$$HFL_C(t,u) = \sum u^d t^s \dim \widehat{HFL}_{d,s}(K).$$

It categorifies the Alexander polynomial in the sense that

$$HFL_C(t,-1) = t^{-\deg \Delta/2} \Delta_C(t).$$

Theorem (-)

Take $\overline{P}^C_g(t,q)$ and let us make a following change in it: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{-2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{1-2\beta}$. We get a polynomial $\widetilde{\Delta}^C_g(t,u)$. Then

$$\widetilde{\Delta}^C_g(t^{-1},u) = t^{-\deg \Delta/2} HFL_C(t,u). \tag{1}$$

Key lemma

Suppose that a cochain complex $\mathcal{C}$ has a filtration $\mathcal{C}_k, \quad k \geq 0$ and an injective operator U of homological degree 2 acting on it such that
1)$U(\mathcal{C}_k) \subset C_{k+1}$ and $U^{-1}(\mathcal{C}_k) \subset \mathcal{C}_{k-1}$
2)$H^*(\mathcal{C}_k/U(\mathcal{C}_k))$ has rank 1 for all k,
Then for all k the rank of $H^*(\mathcal{C}_k/\mathcal{C}_{k+1})$ is at most 1. Let $\{0, \sigma_1, \sigma_2, \ldots\}$ is the set of k such that this rank is 1. Then
3) $H^*(\mathcal{C}_{\sigma_k}/\mathcal{C}_{\sigma_k+1})$ belongs to degree $2k$.

Key lemma cont'd

Let

$$Q(t,q)=\sum_{k=0}^{\infty} q^k t^{\sigma_k},\ \ \overline{Q}(t,q)=Q(t,q)(1-qt).$$

Let us make a following change in $\overline{Q}$: $t^{\alpha}q^{\beta}$ is transformed to $t^{\alpha}u^{2\beta}$, and $-t^{\alpha}q^{\beta}$ is transformed to $t^{\alpha}u^{2\beta-1}$.
4) The result is equal to

$$\sum_{k,n} t^k u^n \dim H^n(\mathcal{C}_k/(\mathcal{C}_{k+1}+U\mathcal{C}_{k-1})).$$

The last result can be reformulated as follows. Consider the complex $\widehat{\mathcal{C}}_k=\mathcal{C}_k/U\mathcal{C}_{k-1}$, then the last homology are the homology of the quotient $\widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k-1}$. The multiplication by $1-qt$ corresponds to the exact sequence

$$0\to \mathcal{C}_{k-1}/\mathcal{C}_k \xrightarrow{U} \mathcal{C}_k/\mathcal{C}_{k+1}\to \widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k+1}\to 0.$$

Conjectures

N. Dunfield, S. Gukov and J. Rasmussen conjectured that all knot homology theories (Khovanov, Heegard-Floer, Khovanov-Rozansky) are parts, or specializations of a unified picture. They conjectured the existence of a triply-graded knot homology theory $\mathcal{H}_{i,j,k}(K)$ with the following properties:

- **Euler characteristic.** Consider the Poincare polynomial

$$\mathcal{P}(K)(a,q,t) = \sum a^i q^j t^k \dim \mathcal{H}_{i,j,k}.$$

  Its value at $t = -1$ equals to the value of the reduced HOMFLY polynomial of the knot K:

$$\mathcal{P}(K)(a,q,-1) = P(K)(a,q).$$

- **Differentials.** There exist a set of anti-commuting differentials d_j for $j \in \mathbb{Z}$ acting in $\mathcal{H}_*(K)$. For $N > 0$, d_N has triple degree $(-2, 2N, -1)$, d_0 has degree $(-2, 0, -3)$ and for $N < 0$ d_N has degree $(-2, 2N, -1 + 2N)$
- **Symmetry.** There exists a natural involution ϕ such that

$$\phi d_N = d_{-N} \phi$$

for all $N \in \mathbb{Z}$.

Let

$$\mathcal{H}^N_{p,k}(K) = \oplus_{iN+j=p} \mathcal{H}_{i,j,k}(K).$$

Conjecture. There exists a homology theory with above properties such that for all $N > 1$ the homology of $(\mathcal{H}^N_*(K), d_N)$ is isomorphic to the $sl(N)$ Khovanov-Rozansky homology. For $N = 0$, $(\mathcal{H}^0_*(K), d_0)$ is isomorphic to the Heegard-Floer knot homology. The homology of d_1 are one-dimensional.

Consider "stable limit" of torus knots $T_{n,m}$ at $m \to \infty$.

$$P_s(T_n) = \lim_{m\to\infty} P_s(T_{n,m}) = \prod_{k=1}^{n-1} \frac{(1-a^2q^{2k})}{(1-q^{2k+2})}.$$

Conjecture The limit homology $\mathcal{H}(T_n) = \lim_{m\to\infty} \mathcal{H}(T_{n,m})$ is a free polynomial algebra with $n-1$ even generators with gradings $(0, 2k+2, 2k)$ and $n-1$ odd generators with gradings $(2, 2k, 2k+1)$, therefore

$$\mathcal{P}_s(T_n) = \prod_{k=1}^{n-1} \frac{(1+a^2q^{2k}t^{2k+1})}{(1-q^{2k+2}t^{2k})}.$$

We denote the odd generators by $\xi_1, \dots, \xi_{n-1}$, and even generators by $e_1, \dots, e_{n-1}$.
The differentials send ξ_k to some polynomials in e_m, and they are extended to the whole algebra by the Leibniz rule. Taking into account the gradings, one can uniquely guess the equations

$$d_{-n}(\xi_k) = \delta_{k,n}, d_0(\xi_k) = e_{k-1}, d_1(\xi_k) = e_k.$$

The construction of the higher differentials is less restricted by the grading, however for small degrees one has no choice but to define

$$d_2(\xi_2) = e_1^2, d_2(\xi_3) = e_1 e_2, d_3(\xi_3) = e_1^3.$$

Example. Knot $T_{3,4}$ (link of E_6):

$$\begin{tikzcd}
 & & \xi_1\xi_2 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\xi_1 \arrow[d] \arrow[dr] & \xi_2 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & e_1\xi_1 \arrow[dl] \arrow[dr] & e_1\xi_2 - e_2\xi_1 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & e_1^2\xi_1 \arrow[dl] \arrow[d] \\
1 & e_1 & e_2 & e_1^2 & e_1^3
\end{tikzcd}$$

Vertical lines correspond to the differential d_0, its homology has dimension 5, as expected for Heegard-Floer homology.