Khovanov Homology and Torus Knots

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Knot invariants

Polynomial invariants

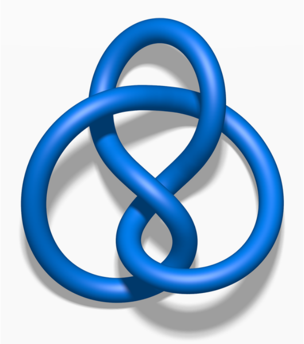
Khovanov homology

Knot invariants

What is a knot?

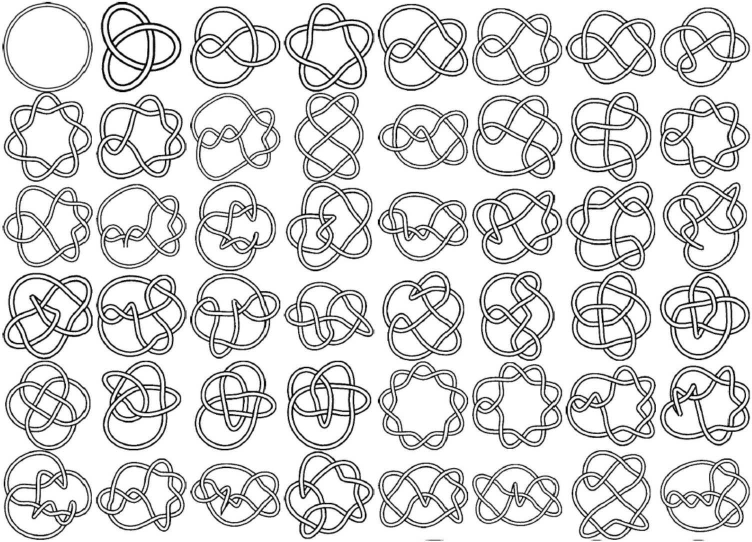
A link is a closed 1-dimensional submanifold in the sphere S^3. If it has one connected component, it is called a knot. Using stereographic projection, one can place knots and links in $\mathbb{R}^3$.

Two knots are equivalent if one can be continuously deformed into another without tearing their strands.



Knot invariants

Examples of knots

 [1]

[1] Rolfsen's knot table, `www.katlas.org`

Knot invariants

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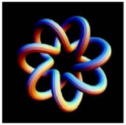
- Quantum computers: quantum entanglement of states can be described by knots
- Physics: Chern-Simons theory, string theory

Knot invariants

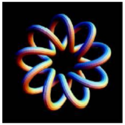
Torus knots

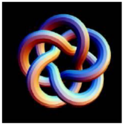

T(3,2)

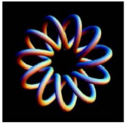

T(5,2)

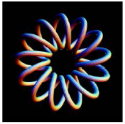

T(7,2)

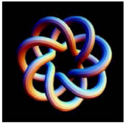

T(4,3)


T(9,2)

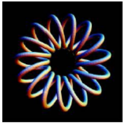

T(5,3)

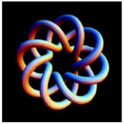

T(11,2)

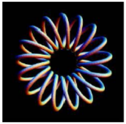

T(13,2)

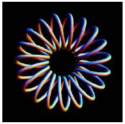

T(7,3)

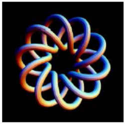

T(5,4)

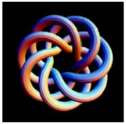

T(15,2)

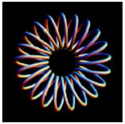

T(8,3)

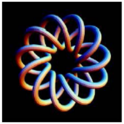

T(17,2)


T(19,2)


T(10,3)


T(7,4)


T(21,2)


T(11,3)

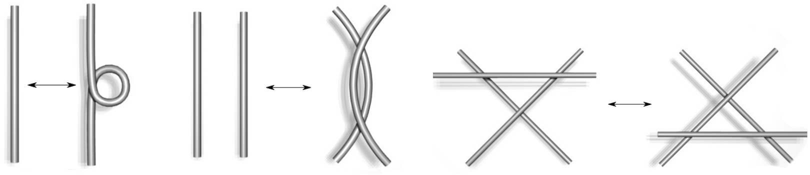
Knot invariants

Reidemeister's theorem

To determine if two pictures describe topologically equivalent knots, one can use the following theorem.

Theorem (K. Reidemeister)

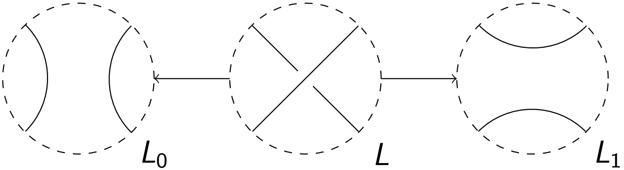
Two knots are topologically equivalent if and only if their projections can be related by a sequence of the transformations of the following three types:



Polynomial invariants

Kauffman bracket

Each crossing on the knot diagram can be resolved in two different ways:



Kauffman bracket is defined by the following equations:

$$\langle \emptyset \rangle = 1,\ \langle O \sqcup L \rangle = (q + q^{-1})\langle L \rangle,\ \langle L \rangle = \langle L_0 \rangle - q\langle L_1 \rangle,$$

where O denotes the trivial knot.

Polynomial invariants

Jones polynomial

Define the Jones polynomial by the formula

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle,$$

where n_+ and n_- denote the number of positive and negative crossings.

Theorem (V. Jones, L. Kauffman)

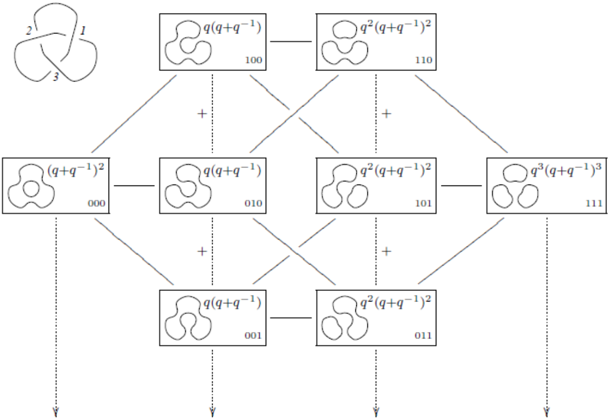
Jones polynomial is preserved by the Reidemeister moves and hence depends only on the topological type of a knot.

Corollary

If two knots have different Jones polynomials, they are not topologically equivalent.

Polynomial invariants

Cube of resolutions



(1)

$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

$$= q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\quad) = q^2+q^6-q^8.$$

Polynomial invariants

Other quantum invariants

Using representation theory of quantum groups and Hecke algebras, one can define more general knot invariants. Given a knot K and a Young diagram (partition) λ, one can define the *λ-colored HOMFLY invariant* $P_\lambda(K)$ depending on two variables a and q.

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- For $a = q^N$ the invariant specializes to the *Reshetikhin-Turaev invariant* for quantum $\mathfrak{sl}_N$.
- In particular, for $a = q^2$ and $\lambda = \square$ one gets the Jones polynomial.

Polynomial invariants

Positivity

Theorem (P. Etingof, E. G., I. Losev, 2013)

For any torus knot $T(m,n)$ and any Young diagram λ the colored HOMFLY invariant $P_\lambda(T(m,n))$ can be expanded in $(-a)$ and q with nonnegative *coefficients.*

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- The nonnegativity fails for the Jones polynomial. Indeed, a change from a to $(-a)$ makes no sense after the substitution $a = q^2$.
- Rosso and Jones proved a formula presenting $P_\lambda(T(m,n))$ as a certain alternating sum. The positivity is not clear from this formula.

Polynomial invariants

Idea of proof

The proof uses the representation theory of *rational Cherednik algebras*. Suppose that λ is a partition of size d.

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- Consider its irreducible representation $L_{m/n}(n\lambda)$ with "lowest weight" $n\lambda$.
- We prove that the character of $L_{m/n}(n\lambda)$ matches the Rosso-Jones formula for the HOMFLY polynomial

As a result, the HOMFLY coefficients can be described as dimensions of certain graded pieces in $L_{m/n}(n\lambda)$, therefore they are nonnegative.

Algebra for the unknot

Consider an algebra A with two generators $1, x$ and the following multiplication:

$$1 \cdot 1 = 1,\ 1 \cdot x = x \cdot 1 = x,\ x \cdot x = 0.$$

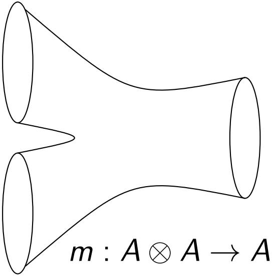
Multiplication can be considered as a map $m : A \otimes A \to A$. One can also define a *comultiplication* $\mu : A \to A \otimes A$ by the equation:

$$\mu(1) = 1 \otimes x + x \otimes 1,\ \mu(x) = x \otimes x.$$

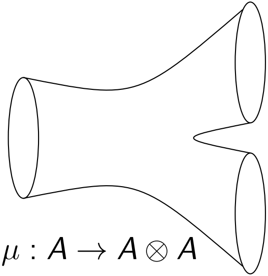
Algebra A is naturally graded, and the maps m and μ behave well with respect to the grading.

Algebra for the unknot cont'd

One can associate the algebra A to a circle and interpret the maps m and μ as pair-of-pants cobordisms between circles:



$m : A \otimes A \to A$



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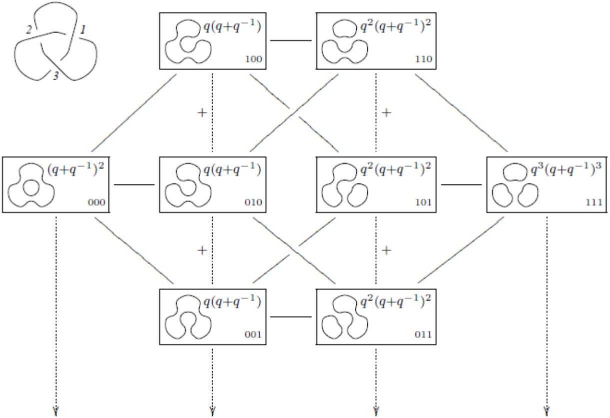
Cube of resolutions revisited



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From cube to chain complex

Let us replace a resolution with k circles with the space $A^{\otimes k}$ in each vertex of a cube. Let C_i be a direct sum of such spaces over vertices with i units. Every edge corresponds either to merging or to splitting two circles, and we can replace it by $\pm m : A \otimes A \to A$ or by $\pm \mu : A \to A \times A$. Therefore edges provide maps $d_i : C_i \to C_{i+1}$.

Lemma (M. Khovanov)

One can choose $\pm$ signs such that $d_i d_{i+1} = 0$, so one gets a chain complex:

$$\dots \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} \dots$$

For the trefoil on previous slide:

$$0 \to A \otimes A \to A \oplus A \oplus A \to (A \otimes A) \oplus (A \otimes A) \oplus (A \otimes A) \to A \otimes A \otimes A \to 0$$

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Khovanov homology was proven to be a powerful tool, much stronger than Jones polynomial. Rasmussen used it to prove Milnor conjecture, and Kronheinmer and Mrowka proved that Khovanov homology detects the unknot.

Torus knots: conjecture

Despite the elementary definition, it is quite hard to compute Khovanov homology. For example, a general formula for the Khovanov homology of torus knots is not known.

Theorem (M. Stošić)

There exists a well defined limit $\mathrm{Kh}(n,\infty)$ *of the Khovanov homology of torus knots* $T(n,m)$ *at* $m\to\infty$*.*

Conjecture (E.G., A. Oblomkov, J. Rasmussen)

Consider a polynomial algebra $\mathcal{H}_n$ *with commuting generators* $x_0,\ldots,x_{n-1}$ *and anticommuting generators* $\xi_0,\ldots,\xi_{n-1}$ *and a differential given by the formula:*

$$D(x_i)=0,\ \ D(\xi_i)=\sum_{j=0}^{i}x_jx_{i-j}.$$

Then $Kh(n,\infty)=H^*(\mathcal{H}_n,D)$

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- The relation $a = q^2$ can be lifted to knot homology as following: there exists a differential D which cancels generators of degree $(-a)$ with the generators of degree q^2. Using Cherednik algebra, one can write a candidate for such D.
- In the stable limit $m \to \infty$ Khovanov-Rozansky homology of (n, ∞) torus knot is isomorphic to $\mathcal{H}_n$.

Evidence for the conjecture

Bar-Natan and Morisson developed a program computing Khovanov homology. It can handle torus knots up to $\min(m, n) = 8$.

- In "stable range", the bigraded ranks of Khovanov homology agree with the algebraic model in all examples.
- The ranks also agree for homology with $\mathbb{Z}_2$ coefficients.
- The algebraic model exhibits expected torsion. For example, there exists a $\mathbb{Z}_p$ torsion in the homology of (p, ∞) torus knot for any prime p.
- The algebraic model satisfies various structural properties known or conjectured for Khovanov homology.
- One can prove that for large n the ranks of the homology in the algebraic model are described by the Rogers-Ramanujan identity.

Thank you