Torus Knots and q, t-Catalan Numbers

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Outline

q, t-Catalan numbers

Compactified Jacobians

Arc spaces on singular curves

Rational Cherednik algebras

Knot homology

q, t-Catalan numbers

Catalan numbers

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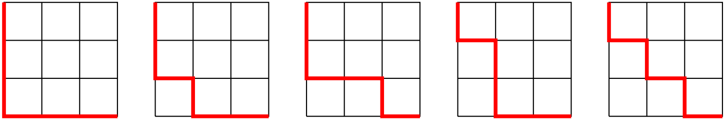
Catalan numbers appear in many combinatorial problems (see "Catalan addendum" by R. P. Stanley). Most important for the present talk: the Catalan number is the number of **Dyck paths**, i. e. lattice paths in $n \times n$ square that never go above the diagonal:



q, t-Catalan numbers

Generalizations

Let m and n be two coprime numbers. The number of lattice paths in $m \times n$ rectangle that never go above the diagonal equals to

$$c_{m,n} = \frac{(m+n-1)!}{m!n!} = \frac{1}{m+n}\binom{m+n}{n}$$

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When $m = n+1$, we get Catalan numbers. When $m = kn+1$, we get Fuss-Catalan numbers (paths in $n \times kn$ rectangle). Finally, the number of lattice paths in $m \times n$ rectangle with k marked corners equals to

$$S_{m,n,k} = \frac{(m+n-k-1)!}{n \cdot m \cdot k! \cdot (m-k-1)!(n-k-1)!}$$

When $m = n+1$, $S_{n,n+1,k}$ is the number of k-dimensional faces of the n-dimensional associahedron.

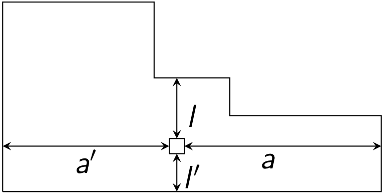
q, t-Catalan numbers

q, t-Catalan numbers: definition

In 1996 A. Garsia and M. Haiman introduced the following bivariate function:

$$C_n(q,t) = \sum_{|\lambda|=n} \frac{t^{2\sum l} q^{2\sum a} (1-t)(1-q) \prod^{0,0} (1-q^{a'} t^{l'}) \sum q^{a'} t^{l'}}{\prod (q^a - t^{l+1})(q^{a+1} - t^l)}.$$

Here l, l', a, a' denote the lengths of leg, co-leg, arm and co-arm of a box in a Young diagram λ, and $\prod^{0,0}$ indicates that the box $(0,0)$ does not contribute to the product.



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- They "deform" the Catalan numbers: $c_n(1,1) = c_n$
- More precisely, there is an interesting degeneration:

$$c_n(q, q^{-1}) = q^{-\binom{n}{2}} \frac{[2n!]_q}{[n!]_q[(n+1)!]_q},$$

where

$$[k!]_q := \frac{(1-q)\cdots(1-q^k)}{(1-q)^k}.$$

q, t-Catalan numbers

q, t-Catalan numbers: examples

Another interesting degeneration: $c_n(q,1)$ is a sum over all Dyck paths weighted by the area between a path and the diagonal:

$$c_n(q,1) = \sum_D q^{\binom{n}{2} - |D|}.$$

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Examples:

$$c_1(q,t) = 1, \quad c_2(q,t) = q + t,$$

$$c_3(q,t) = q^3 + q^2 t + qt + t^2 q + t^3.$$

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$$c_3(q,t) = q^3 + q^2 t + qt + t^2 q + t^3.$$

We have 1 path with $|D| = 0$, 1 path with $|D| = 1$, 2 paths with $|D| = 2$ and 1 path with $|D| = 3$:

$$c_3(q,1) = 1 + 2q + q^2 + q^3.$$

q, t-Catalan numbers

q, t-Catalan numbers: Hilbert scheme

$Hilb^n(\mathbb{C}^2)$ – Hilbert scheme of n points on $\mathbb{C}^2$.

T – tautological bundle of rank n on it

$Z = Hilb^n(\mathbb{C}^2, 0)$ – Hilbert scheme of n points supported at 0

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Theorem

(M. Haiman)

1. $H^i(Z, \Lambda^n T) = 0$ *for* $i > 0$
2. $\chi_{(C^*)^2}(Z, \Lambda^n T) = c_n(q, t)$.

Corollary

The function $c_n(q, t)$ *is a character of* $(C^*)^2$ *action on* $H^0(Z, \Lambda^n T)$, *so it is a polynomial with nonnegative coefficients.*

Compactified Jacobians

Construction

Consider a plane curve singularity $C = C_{m,n} = \{x^m = y^n\}$, m and n coprime. The **compactified Jacobian** of C is the moduli space of rank 1 degree 0 torsion-free sheaves on a projective rational curve with unique singularity C.

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Local description: consider $R = \mathbb{C}[[t^m, t^n]]$ – the ring of functions on C. The JC is the moduli space of R-modules inside $\overline{R} = \mathbb{C}[[t]]$.

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If $(m, n) = (2, 3)$, we get a Jacobian of a cuspidal cubic. It is a cuspidal cubic itself, *topologically* $JC \simeq \mathbb{P}^1$.

Compactified Jacobians

Cell decomposition

Theorem

(E. G., M. Mazin, 2011) The compactified Jacobian of $C_{m,n}$ *admits an affine cell decomposition. The cells are parametrized by the lattice paths in* $m \times n$ *rectangle below the diagonal, and the dimension of such a cell is given by the formula:*

$$\dim \Sigma_D = \frac{(m-1)(n-1)}{2} - h_+(D),$$

where

$$h_+(D) = \sharp \left\{ c \in D \;\middle|\; \frac{a(c)}{l(c)+1} \le \frac{m}{n} < \frac{a(c)+1}{l(c)} \right\}.$$

Compactified Jacobians

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It turns out that in the case $(m, n) = (n, n+1)$ the statistics h_+ is related to q, t-Catalan numbers.

Theorem

(A. Garsia, M. Haiman, J. Haglund) The q, t-Catalan numbers admit a combinatorial interpretation:

$$c_n(q,t) = \sum_D q^{\binom{n}{2} - |D|} t^{h_+(D)}.$$

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Corollary

The Poincaré polynomial of $JC_{n,n+1}$ equals to

$$P(t) = \sum_D t^{\binom{n}{2} - h_+(D)} = t^{\binom{n}{2}} c_n(1, t^{-1}) = t^{\binom{n}{2}} c_n(t^{-1}, 1) = \sum_D t^{|D|}.$$

Arc spaces on singular curves

Construction

Let us parametrize the singularity as $(x, y) = (t^n, t^m)$ and perturb this parametrization:

$$x(t) = t^n + u_2 t^{n-2} + \ldots + u_n, \quad y(t) = t^m + v_2 t^{m-2} + \ldots + v_m.$$

We regard u_i and v_i as free parameters, and require that the perturbed parameterization satisfies the equation of our curve:

$$A_{m,n} = \mathbb{C}[u_i, v_i]/(t^n + u_2 t^{n-2} + \ldots + u_n)^m = (t^m + v_2 t^{m-2} + \ldots + v_m)^n.$$

We require the identity of $x(t)^m$ and $y(t)^m$ in all orders of t-expansion.

Arc spaces on singular curves

Construction cont'd

Theorem

(B. Fantechi, L. Göttsche, D. van Straten)

- $A_{m,n}$ *is a ring of functions on a 0-dimensional complete intersection*

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Theorem

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- $A_{m,n}$ *is a ring of functions on a 0-dimensional complete intersection*
- $\dim A_{m,n} = \frac{(m+n-1)!}{m!n!}$
- $A_{m,n}$ *is naturally graded and its Hilbert series equals to*

$$H(q) = \frac{[(m+n-1)!]_q}{[m!]_q[n!]_q}.$$

Conjecture(L. Göttsche) The ring $A_{m,n}$ is isomorphic to the cohomology ring of $JC_{m,n}$.

Rational Cherednik algebras

Dunkl operators

Consider the polynomial ring $\mathbb{C}[x]$ in variables x_i with condition $\sum x_i = 0$.

Definition

The Dunkl operator on $\mathbb{C}[x]$ is defined by a formula

$$D_i f = \frac{\partial f}{\partial x_i} - c \sum_{j \neq i} \frac{f - s_{ij} f}{x_i - x_j},$$

where s_{ij} is an transposition of x_i and x_j.

Lemma

$$[D_i, D_j] = 0.$$

Dunkl operators play important role in the study of the quantum rational Calogero-Moser system.

Rational Cherednik algebras

Finite-dimensional representations

Dunkl operators, multiplications by x_i and permutations produce the representation of the rational Cherednik algebra H_c on $\mathbb{C}[x]$.

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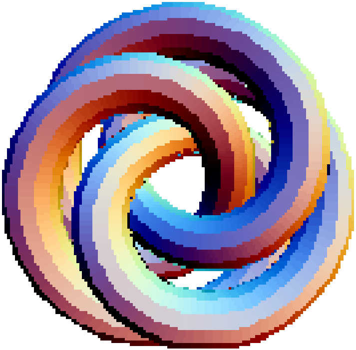
Knot homology

Torus knots

The intersection of singularity $C_{m,n}$ with a small 3-sphere centered at the origin is the (m, n) torus knot.



Knot homology

Conjecture

In 2005 M. Khovanov and L. Rozansky introduced triply graded knot homology theory, whose (bigraded) Euler characteristic gives the HOMFLY-PT polynomial.

Conjecture (E.G., A. Oblomkov, J. Rasmussen, V. Shende) The triply graded Khovanov-Rozansky homology of (m, n) torus knot is isomorphic to

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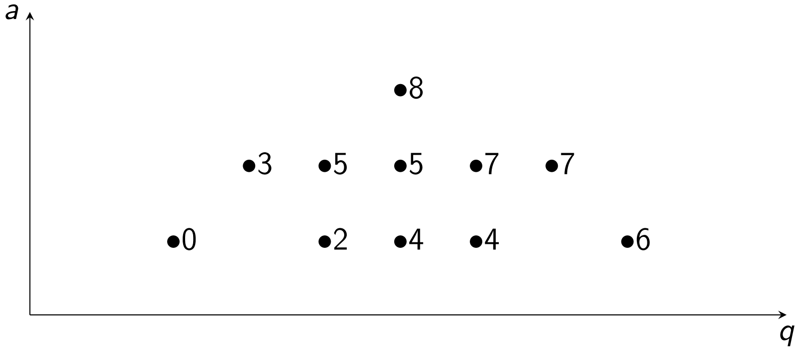
For $k = 0$ we get $(L_{\frac{m}{n}})^{S_n}$.

Knot homology

Example: (3,4) torus knot

For (3,4) torus knot the triply graded homology were computed by S. Gukov, N. Dunfield and J. Rasmussen:



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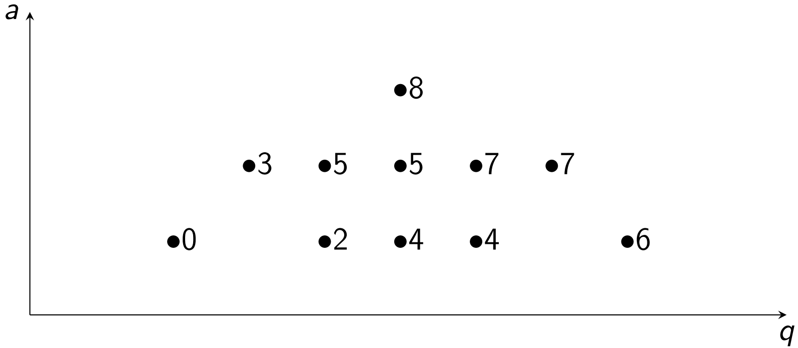

In the lowest a-grading we see slightly regraded $c_3(q,t)$.

Thank you.