# Khovanov-Rozansky homology and the flag Hilbert scheme (joint with Andrei Neguț, Jacob Rasmussen and Paul Wedrich )

Eugene Gorsky
University of California, Davis

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# Khovanov-Rozansky homology

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, define $R - R$ bimodules $B_i = R \otimes_{R^{s_i}} R$.

## Theorem (Rouquier)

*The complexes* $T_i = [B_i \to R]$, $T_i^{-1} = [R \to B_i]$ *satisfy braid relations up to a homotopy:*

$$T_i \otimes T_i^{-1} \sim \mathbf{1} = R,\ T_i \otimes T_j \simeq T_j \otimes T_i\ (|i - j| > 1),$$

$$T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1}.$$

As a consequence, to every braid $\beta$ one can associate a complex (which we will also denote by $\beta$) in the *homotopy category of Soergel bimodules* $K(\mathrm{SBim}_n)$.

## Theorem (Khovanov)

*The* Khovanov-Rozansky homology $\mathrm{HHH}(\beta) = R\,\mathrm{Hom}(\mathbf{1}, \beta)$ *is a topological invariant of the closure of* $\beta$.

# Flag Hilbert scheme

The Hilbert scheme of points in $\mathbb{C}^2$ is defined as the moduli space of codimension $n$ ideals in $\mathbb{C}[x, y]$. We define an algebraic variety $\mathrm{FHilb}^n$ as the moduli space of flags of ideals $\{\mathbb{C}[x, y] \supset I_1 \supset \ldots \supset I_n\}$ such that $\dim \mathbb{C}[x, y]/I_k = k$ for all $k$ and $\mathbb{C}[x, y]/I_k$ is set-theoretically is supported on $\{y = 0\}$.

It can be also defined as the space of triples $(X, Y, v)$ where $X$ is a lower-triangular $n \times n$ matrix, $Y$ is a strictly lower-triangular $n \times n$ matrix such that $[X, Y] = 0$, $v \in \mathbb{C}^n$ satisfies a certain stability condition, modulo the equivalence relation $(X, Y, v) \sim (gXg^{-1}, gYg^{-1}, gv)$.

## Example

For $n = 2$ one can pick $v = (1, 0)$,

$$X = \begin{pmatrix} x_1 & 0 \\ z & x_2 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}, \; w(x_1 - x_2) = 0.$$

# Main conjecture

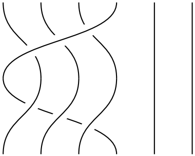
There is a pair of adjoint functors

$$\iota_* : K(\mathrm{SBim}_n) \leftrightarrow D(\mathrm{FHilb}^n) : \iota^*$$

satisfying the following:

- $\iota^*(A \otimes B) = \iota^*(A) \otimes \iota^*(B)$
- $\iota_*(\iota^*(A) \otimes B) = \iota_*(B \otimes \iota^*(A)) = A \otimes \iota_*(B)$
- $\iota_*(\mathbf{1}) = \mathcal{O}$
- $\iota^*(\mathcal{L}_i) = L_i$, where $\mathcal{L}_i$ are certain line bundles on $\mathrm{FHilb}^n$ and

$L_i =$



$i$

- $\iota_*(\beta)$ coincides with the Oblomkov-Rozansky homology

# Application

One can compute Khovanov-Rozansky homology of some links using algebraic geometry:

$$\mathrm{HHH}\left(\prod L_i^{a_i}\right) = \mathrm{Hom}\left(\mathbf{1}, \bigotimes L_i^{a_i}\right) = \mathrm{Hom}\left(\iota^*(\mathcal{O}), \bigotimes L_i^{a_i} \otimes \mathbf{1}\right) =$$

$$\mathrm{Hom}\left(\mathcal{O}, \iota_*\left(\bigotimes L_i^{a_i} \otimes \mathbf{1}\right)\right) = \mathrm{Hom}\left(\mathcal{O}, \bigotimes \mathcal{L}_i^{a_i} \otimes \iota_*(\mathbf{1})\right) =$$

$$\mathrm{Hom}\left(\mathcal{O}, \bigotimes \mathcal{L}_i^{a_i}\right) = H^*\left(\mathrm{FHilb}^n, \bigotimes \mathcal{L}_i^{a_i}\right)$$

In some cases, this is confirmed by recent computations of Elias, Hogancamp and Mellit.

## Theorem (Hogancamp,Mellit)

*The Poincaré polynomial for the* $(a = 0)$ *part of* HHH *for the torus knot* $T(m, n)$ *equals* $(P_{m,n} \cdot 1, e_n)$*, where* $P_{m,n}$ *is the generator of the Elliptic Hall Algebra.*

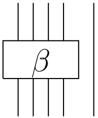
G., Neguț, Carlsson, Mellit gave a combinatorial description of this polynomial.

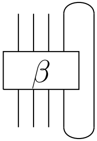
# Inductive step

Moreover, we conjecture the existence of commutative diagrams:

$$\begin{tikzcd}
D(\mathsf{FHilb}^{n+1}) \arrow[r, shift left, "\iota^*"] \arrow[d, shift left, "\pi_*"] & K(\mathsf{SBim}_{n+1}) \arrow[l, shift left, "\iota_*"] \arrow[d, shift left, "\mathrm{Tr}"] \\
D(\mathsf{FHilb}^{n}) \arrow[r, shift left, "\iota^*"] \arrow[u, shift left, "\pi^*"] & K(\mathsf{SBim}_{n}) \arrow[l, shift left, "\iota_*"] \arrow[u, shift left, "i"]
\end{tikzcd}$$

where

$i(\beta) =$ 


$\mathrm{Tr}(\beta) =$ 


Note that $\mathrm{HHH}(\beta) = \mathrm{Tr}^n(\beta)$.

# Projective bundles, geometrically

Geometrically, we can factor the projection $\pi : \mathrm{FHilb}^{n+1} \to \mathrm{FHilb}^n$ into two steps:

$$\mathrm{FHilb}^{n+1} \xrightarrow{p} \mathrm{FHilb}^n \times \mathbb{C} \xrightarrow{q} \mathrm{FHilb}^n .$$

## Theorem

*The fibers of the map p are projective spaces of various dimensions. More precisely,*

$$\mathrm{FHilb}^{n+1} = \mathbb{P}(\mathcal{E}_n^\vee),\ \mathcal{E} = [q^*\mathcal{V}_n \xrightarrow{X - x_{n+1}} q^*\mathcal{V}_n],$$

*where $\mathcal{V}_n$ is a certain explicit sheaf on $\mathrm{FHilb}^n$ with an endomorphism $X$. Furthermore, $\mathcal{L}_{n+1} = \mathcal{O}(1)$ for this projective bundle.*

## Example

The projection $\mathrm{FHilb}^2 \to \mathrm{FHilb}^1 \times \mathbb{C}$ sends $(X, Y)$ to $(x_1, x_2)$. For $x_1 \neq x_2$, the fiber is a point. For $x_1 = x_2$, the fiber is $\mathbb{P}^1$.

# Projective bundles, algebraically

How to write this algebraically? The category $D(\mathrm{FHilb}^{n+1})$ is generated by the objects of the form $p^*(A) \otimes \mathcal{L}_{n+1}^k$ where $A \in D(\mathrm{FHilb}^n)$. They satisfy certain relations similar to the Koszul complex, relating the powers of $\mathcal{O}(1)$ on $\mathbb{P}^n$.[1]

For $k > 0$, one has

$$p_*(\mathcal{L}_{n+1}^k) = S^k \mathcal{E}_n.$$

## Lemma

$$\pi_*(\mathcal{L}_{n+1}^k) = q_* p_*(\mathcal{L}_{n+1}^k) = q_*(S^k \mathcal{E}_n) \simeq$$

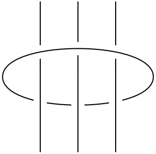
$$q_*[q^* \wedge^k \mathcal{V}_n \to q^* \wedge^{k-1} \mathcal{V}_n \otimes q^* \mathcal{V}_n \to \ldots \to q^* S^k \mathcal{V}_n] \simeq$$

$$[\wedge^k \mathcal{V}_n \to \ldots \to \mathbb{S}^{k-1,1} \mathcal{V}_n \to S^k \mathcal{V}_n].$$

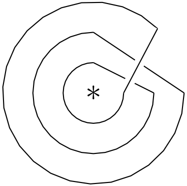
[1] In fact, this is closely related to categorical diagonalization from Matt Hogancamp's talk

# Projective bundles, topologically

How to describe all this information in terms of braids?

$$\mathrm{Tr}(L_{n+1}) = \iota^* \mathcal{V}_n =$$ 

From above, this looks like a circle wrapping other $n$ strands.
Similarly, $\mathrm{Tr}(L_{n+1}^k)$ from above looks like a $k$-stablized unknot:




To relate it to $\mathcal{V}_n$ as in the previous slide, we use the machinery of *annular Khovanov-Rozansky homology* developed by Queffelec and Rose.

Thank you