Rational Catalan Combinatorics

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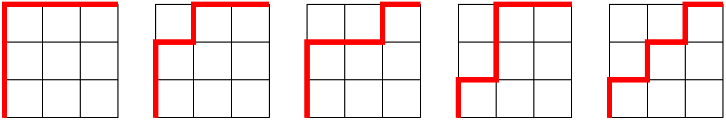
Counting Dyck paths

Catalan numbers

The Catalan number is the number of Dyck paths, that is, lattice paths in $n \times n$ square that never cross the diagonal:



Named after Belgian mathematician Eugène Charles Catalan (1814–1894), probably discovered by Euler.

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.$$

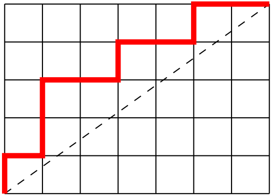
First values: $1, 2, 5, 14, 42, 132 \ldots$

Catalan numbers have many different combinatorial interpretations (counting trees, triangulations of a polygon, noncrossing partitions etc).

Counting Dyck paths

Rational Catalan numbers

An (m, n) Dyck path is a lattice path in $m \times n$ rectangle which never crosses the diagonal. Here's an example of a $(5, 7)$ Dyck path:



Let $c_{m,n}$ be the number of (m, n) Dyck paths. Clearly,

$$c_{m,n} = c_{n,m} \text{ and } c_{n,n+1} = c_{n,n} = c_n$$

If $GCD(m, n) = 1$, then

$$c_{m,n} = \frac{(m+n-1)!}{m!n!}$$

Counting Dyck paths

Non-coprime case

The formula for $c_{m,n}$ in the non-coprime case is much more complicated.

Theorem (Bizley,Grossman)

Given a pair of coprime integers (m, n), the generating function for $c_{dm,dn}$ has the form:

$$\sum_{d=0}^{\infty} c_{dm,dn} z^d = \exp\left(\sum_{d=1}^{\infty} \frac{(km+kn-1)!}{(km)!(kn)!} z^k\right).$$

Remark

For $k > 1$, $\frac{(km+kn-1)!}{(km)!(kn)!}$ usually is not an integer. However, these rational numbers together with the coefficients of the Taylor series for $\exp(x)$ blend into a series with integer coefficients.

Equivalent descriptions

Simultaneous cores

From now on, we focus on $c_{m,n}$ for coprime (m, n).

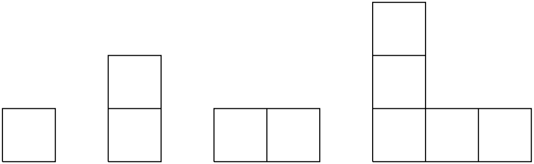
A Young diagram is called an *n–core*, if neither of the hook-lengths of its cells is divisible by n.

Theorem (J. Anderson)

For coprime m, n there is a bijection between the set of (m, n) Dyck paths and the set of (m, n)-cores (that is, partitions with are both m-cores and n-cores). In particular, the number of (m, n)-cores is finite and equals $c_{m,n}$.

Here's a list of $(3, 4)$-cores:

$\emptyset$



Equivalent descriptions

Semigroup modules

Theorem (G., Mazin)

There is a bijection between (m, n) Dyck paths and subsets $M \subset \mathbb{Z}$ such that $\min(M) = 0$, $M + n \subset M$, $M + m \subset M$. Such subsets can be interpreted as (semi)-modules over the integer semigroup generated by m and n.

Example

For $(m, n) = (3, 4)$ one has the following modules:

$$0, 3, 4, 6, \ldots, \quad 0, 3, 4, 5, 6, \ldots, \quad 0, 2, 3, 4, 5, 6, \ldots,$$

$$0, 1, 3, 4, 5, 6, \ldots, \quad 0, 1, 2, 3, 4, 5, 6, \ldots$$

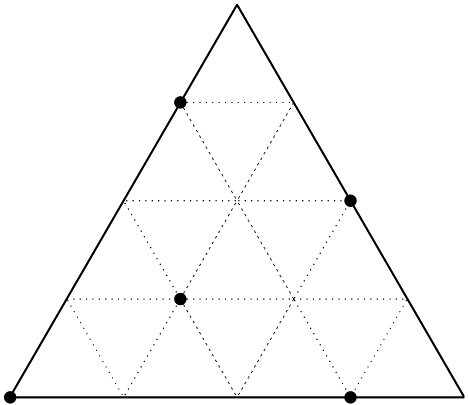
Moyano-Fernández and Uliczka studied these modules from the viewpoint of commutative algebra (generators, syzigies etc).

Equivalent descriptions

Lattice points in a simplex

Theorem (G.,Mazin,Vazirani; P. Johnson)

There is a bijection between the set of (m, n) Dyck paths and the set of lattice points in a certain $(n-1)$-dimensional simplex of size m.



Equivalent descriptions

Theorem (Berest,Etingof,Ginzburg)

The rational Cherednik algebra has a unique finite-dimensional representation $L_{m/n}$, and

$$\dim L_{m/n} = m^{n-1},\ \dim(L_{m/n})^{S_n} = \frac{(m+n-1)!}{m!n!} = c_{m,n}.$$

Problem: find a basis in $(L_{m/n})^{S_n}$ labeled by (m,n) Dyck paths.

Theorem (Gordon)

For $m = n+1$, the space $L_{m/n}$ is isomorphic to the space of diagonal harmonics*:*

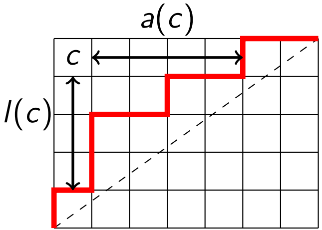
$$L_{\frac{n+1}{n}} \simeq DH_n = \frac{\mathbb{C}[x_1\dots,x_n,y_1\dots,y_n]}{\mathbb{C}[x_1\dots,x_n,y_1\dots,y_n]^{S_n}_+}.$$

Statistics

There are many interesting statistics on rational Dyck paths:

- Area above the Dyck path
- Area of the corresponding (m, n)–core (studied by Armstrong, Johnson and others)
- dinv statistics (motivated by the work of Garsia, Haglund, Haiman on q, t-Catalan numbers)

$$\mathrm{dinv}(D) = \sharp\left\{c : \frac{a(c)}{l(c)+1} < \frac{m}{n} < \frac{a(c)+1}{l(c)}\right\}$$



Statistics

Dinv and geometry

It turns out that dinv statistics has an important geometric meaning. Let C be a plane curve singularity defined by the equation $\{x^m = y^n\}$, and let $X_{m,n}$ be the Hilbert scheme of N points on C for N large enough. The variety $X_{m,n}$ is also known as *compactified Jacobian* of C or *affine Springer fiber*.

Theorem (G.,Mazin)

The algebraic variety $X_{m,n}$ has a paving by affine cells Σ_D, which are naturally labeled by the (m,n) Dyck paths D. The dimension of such a cell equals

$$\dim \Sigma_D = \frac{(m-1)(n-1)}{2} - \mathrm{dinv}(D).$$

Conjectures

Symmetry conjecture

Let $\delta_{m,n} = \frac{(m-1)(n-1)}{2}$. Define

$$c_{m,n}(q,t) = \sum_D q^{\delta_{m,n} - \mathrm{area}(D)} t^{\mathrm{dinv}(D)}.$$

Conjecture

This polynomial is symmetric : $c_{m,n}(q,t) = c_{m,n}(t,q)$.

For $m = n+1$, the polynomial $c_{n,n+1}(q,t)$ is known as q,t-Catalan polynomial, and the symmetry was proved by Garsia and Haglund.

Theorem (Lee,Li,Loehr)

Conjecture holds for $\min(m,n) \leq 4$.

Theorem (G.Mazin)

For $m = kn \pm 1$, *one has* $c_{m,n}(q,1) = c_{m,n}(1,q)$.

Conjectures

Conjecture (G., Neguț)

The polynomial $c_{m,n}(q,t)$ can be computed as follows:

$$c_{m,n}(q,t) = (P_{m,n} \cdot 1, e_n),$$

where $P_{m,n}$ is a certain operator acting on symmetric functions.

The right hand side is manifestly symmetric in q and t, and can be computed as an explicit sum of rational functions over standard Young tableaux of size n (or m).

The operators $P_{m,n}$ generate the so-called *elliptic Hall algebra*, which has been an object of active study in representation theory (by Burban, Feigin, Schiffmann, Tsymbaliuk, Vasserot...).

Further directions

- Generalization to non-coprime case:
  Bergeron, Garsia, Leven, Xin....
- Rational parking functions:
  Armstrong, Loehr, Warrington ...
- Rational associahedra:
  Armstrong, Rhoades, Williams.....
- Connections to LLT polynomials:
  Haglund, Haiman, Loehr, Ulyanov, Remmel; G., Mazin
- Connections to knot invariants:
  G., Neguț, Oblomkov, Rasmussen, Shende....

Thank you