# Singular curves and link invariants

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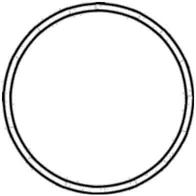
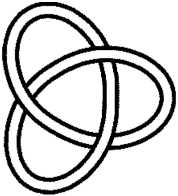
# Singular curves and links

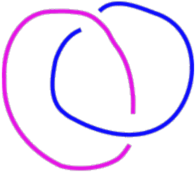
Let $(C, 0)$ be a complex plane curve singularity. Its intersection with a small sphere $S^3$ is a knot or a link $L$.



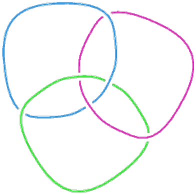
$y = 0$



$x^2 = y^3$



$xy = 0$



$x^3 = y^3$

Image credits: `katlas.org`

# Singular curves and links

Classical question: how to relate the topological invariants of $L$ to the algebro–geometric invariants of $C$?

This problem was studied by Milnor, Eisenbud, Neumann, A'Campo and others. For example, for a unibranched curve one gets an iterated torus knot, and its cabling parameters match the Puiseaux pairs of the singularity.

# Singular curves and links

I will discuss some new developments which can be described (sometimes conjecturally) as follows:

- One constructs a family of moduli spaces defined by the algebraic geometry of $C$
- The Euler characteristics of these spaces match the coefficients of *polynomial invariants* (Alexander and Jones polynomials...) of $L$
- Furthermore, the homology of these spaces are expected to match the *homogical invariants* (Heegaard Floer and Khovanov homologies...) of $L$

# Alexander polynomial

Suppose that $C$ has $r$ irreducible components $C_1, \dots, C_r$, let $\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$ denote their uniformizations. For $g \in \mathbb{C}[[x, y]]$ we define its order on $C_i$ as

$$\nu_i(g) = \operatorname{Ord}_0 g(\gamma_i(t)).$$

For $v \in \mathbb{Z}^r$ consider the space

$$\mathcal{H}(v) := \{g \in \mathbb{C}[[x, y]] : \nu_i(g) = v_i \ \forall i\}.$$

## Theorem (Campillo, Delgado, Gusein-Zade)

*The following identity holds:*

$$\sum_{v \in \mathbb{Z}^r} \chi(\mathcal{H}(v)/\mathbb{C}^*) t^v = \begin{cases} \Delta(t_1, \dots, t_r) & \text{if } r > 1 \\ \Delta(t)/(1 - t) & \text{if } r = 1, \end{cases}$$

*where $\Delta(t)$ is the Alexander polynomial of $L$.*

# Alexander polynomial

For example, consider the Hopf link, corresponding to the singularity $\{xy = 0\}$. A function $g \in \mathbb{C}[x, y]$ has order 0 on one of the components if and only its constant term is nonzero, and hence its order on the second component also equals 0. Therefore

$$\mathcal{H}(0,0) = \{\alpha + \text{higher order terms} | \alpha \neq 0\} \sim \mathbb{C}^*,$$

$$\mathcal{H}(a,0) = \mathcal{H}(0,a) = \emptyset \text{ for } a > 0.$$

Furthermore, for $a, b > 0$ one has

$$\mathcal{H}(a,b) = \{\alpha x^a + \beta y^b + \text{higher order terms} | \alpha, \beta \neq 0\} \sim (\mathbb{C}^*)^2.$$

Therefore

$$\sum_{v \in \mathbb{Z}^2} \chi(\mathcal{H}(v)/\mathbb{C}^*) t^v = 1 = \Delta(t_1, t_2)$$

# HOMFLY-PT polynomial

The HOMFLY-PT polynomial $P(a,q)$ is a more subtle invariant of links. The specializations $P(1,q)$ and $P(q^2,q)$ give the Alexander and Jones polynomials respectively.

### Theorem (Maulik, conj. by Oblomkov–Shende)

*The following identity holds:*

$$P_L(a,q) = (1-q)\sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(C,0)} (1-a)^{m-1} d\chi,$$

*where* $\mathrm{Hilb}^n C$ *is the punctual Hilbert scheme of* $n$ *points on* $C$ *and* $m$ *is the minimal number of generators for an ideal.*

### Remark

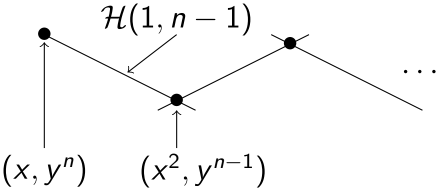
*At* $a=1$ *only principal ideals (with* $m=1$*) survive.*

# HOMFLY-PT polynomial

Let us describe the $n$-th Hilbert scheme of the singularity $\{xy = 0\}$. We have

$$\mathrm{Hilb}^0(C, 0) = \mathrm{Hilb}^1(C, 0) = \{*\}$$

$\mathrm{Hilb}^2(C, 0)$ contains two monomial ideals $(x, y^2)$ and $(y, x^2)$ and a $\mathbb{C}^* = \mathcal{H}(1, 1)/\mathbb{C}^*$ of principal ideals $(\alpha x + \beta y)$, which glue together to $\mathbb{P}^1$. Similarly, $\mathrm{Hilb}^n(C, 0)$ is a chain of $(n-1)$ projective lines:




Note that each of these lines contains $\mathcal{H}(v)/\mathbb{C}^* = \mathbb{C}^*$ for some $v$.

# Heegaard Floer homology

## Theorem (G., Némethi)

*For all $v \in \mathbb{Z}^r$, one has $H_*(\mathcal{H}(v)) \simeq \mathrm{HFL}^-(L, v)$, where $\mathrm{HFL}^-$ denotes a certain version of the Heegaard Floer link homology defined by Ozsváth and Szabó.*

For $r = 1$, this result follows from the earlier work of Hedden, but for $r > 1$ it is new. It provides an effective tool of computing $\mathrm{HFL}^-$ for some links where direct Floer-theoretic methods are hard to apply: for example, for the $(n, n)$ torus link (corresponding to $\{x^n = y^n\}$) the homology are quite subtle.

## Theorem (G., Némethi)

*The Poincaré polynomial of $\mathcal{H}(v)$ is equivalent (up to some change of variables) to the "motivic Poincaré series" of $C$ defined and studied by Campillo–Delgado–Gusein-Zade and Moyano-Zuñiga.*

# Khovanov-Rozansky homology

Khovanov and Rozansky defined a link homology theory "categorifying" HOMFLY-PT polynomial.

## Conjecture (Oblomkov, Rasmussen, Shende)

*The $a = 0$ part of the Poincaré polynomial of the Khovanov-Rozansky homology of $L$ equals*

$$\sum_{n=0}^{\infty} \sum_{i} q^n t^i \dim H_i(\mathrm{Hilb}^n(C, 0)).$$

The conjecture is open even for $r = 1$, since the homology (and the geometry) of $\mathrm{Hilb}^n(C, 0)$ are not known in general. If $C$ has one Puiseaux pair, the answer was computed by Piontkowski, but the Khovanov-Rozansky homology for most torus knots is not known.

# Heegaard Floer homology cont'd

Let us give a more detailed description of the spaces $\mathcal{H}(v)$.

## Lemma

*The space $\mathcal{H}(v)$ is either empty or it is a complement to a hyperplane arrangement.*

Indeed, let

$$J(v) := \{g \in \mathbb{C}[[x, y]] : \nu_i(g) \geq v_i \ \forall i\}.$$

Then $\mathcal{H}(v) = J(v) \setminus \cup_{u \succ v} J(u)$.

## Theorem (Brieskorn,Orlik-Solomon)

*The homology of a complement to a hyperplane arrangement are completely determined by its combinatorics, that is, by the codimensions of intersections of its hyperplanes.*

# Heegaard Floer homology cont'd

Therefore, the answer is determined by the *Hilbert function* of the singularity

$$h(v) := \dim \mathbb{C}[[x, y]]/J(v).$$

A priori, $h(v)$ is an analytic invariant of $C$, but it turns out that it depends only on the topology of $L$.

## Theorem (Moyano-Fernández)

*The function $h(v)$ is determined by the collection of multi-variable Alexander polynomials of L and all its sublinks.*

Furthermore, $h(v)$ determines and is determined by the multi-dimensional semigroup of $C$.

# Heegaard Floer homology cont'd

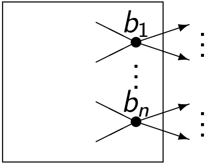
Idea of proof:

1. A sufficiently large surgery of $S^3$ along $L$ is a link of a *rational* surface singularity.
2. (Némethi) Links of rational surface singularities have "easy" Heegaard Floer homology.
3. If a large surgery along $L$ has "easy" homology then there is a spectral sequence with combinatorial $E_2$ page (determined by $\Delta(L)$) and $E_\infty = \mathrm{HFL}^-(L)$.
4. For algebraic links, this spectral sequence collapses and $E_2 = E_\infty$.

Step (3) uses "large surgery theorem" relating $\mathrm{HFL}^-$ for the link and its surgeries. Step (4) follows from the properties of hyperplane arrangements which imply the isomorphism $E_2 \simeq H_*(\mathcal{H}(v)/\mathbb{C}^*)$ and the vanishing of higher differentials – it is the most technical part of the proof.

# Heegaard Floer homology cont'd

Let us prove that the large surgery along the link is a link of rational surface singularity.




Embedded resolution graph of $C$

# Heegaard Floer homology cont'd

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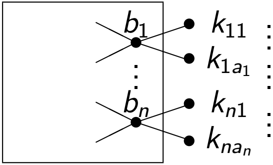


Plumbing graph for $S^3_d(L)$, $k_{ij} = d_{ij} - m_i$, where $m_i$ are the multiplicities of the pullback of $f_{ij}$ on the divisor $E_i$. We can assume $k_{ij} \geq 1$ for all $1 \leq i \leq n$, $1 \leq j \leq a_i$.

# Heegaard Floer homology cont'd

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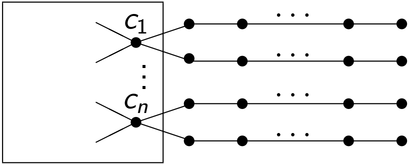


Here's an equivalent plumbing graph of $S^3_d(L)$. If we add extra $(-1)$–vertices, it will become smooth, so $S^3_d(L)$ is a link of a sandwiched singularity.

# Heegaard Floer homology cont'd

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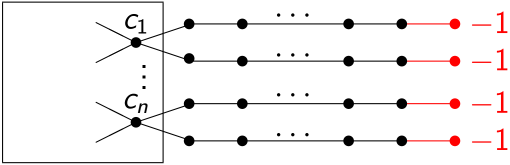


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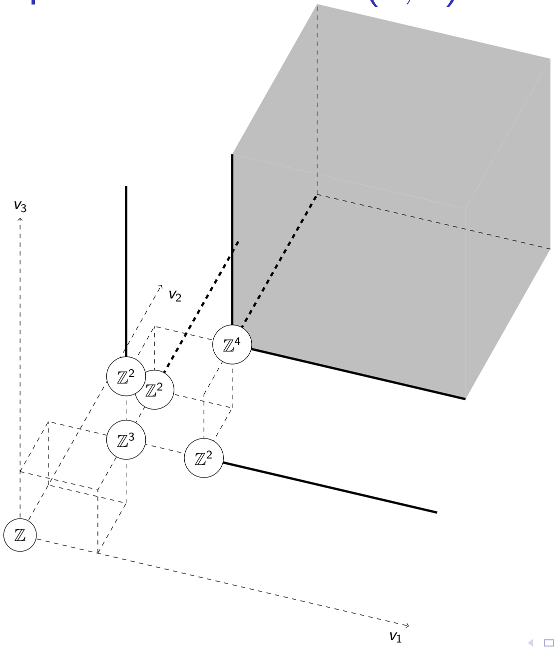
# Example: $HFL^-$ for the $(3,3)$ torus link



Thank you