Algebraic geometry and knot invariants

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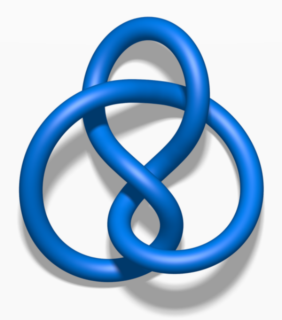
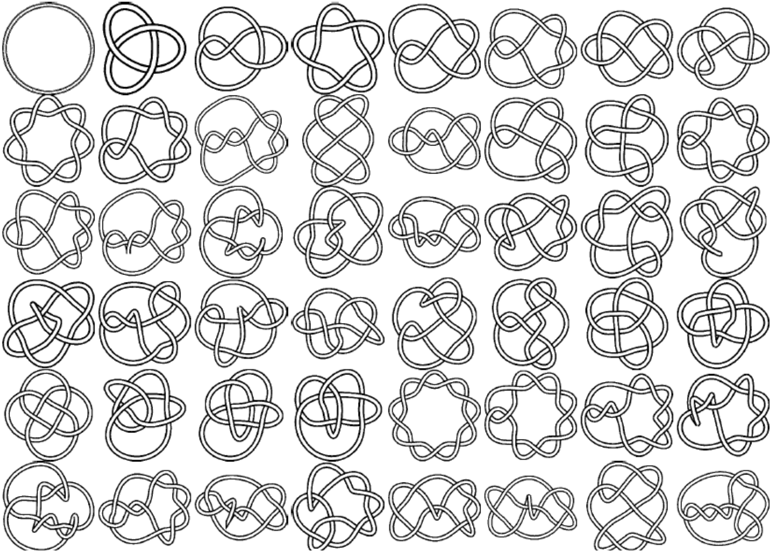
- Knots are complicated, algebraic geometry allows explicit computations
- Ideas from algebraic geometry motivate new, more rigid structures in knot invariants
- Ideas from topology provide new perspective to algebro-geometric problems
- Both topology and algebraic geometry have deep relations to mathematical physics

A link is a closed 1-dimensional submanifold in the sphere S^3. If it has one connected component, it is called a knot. Using stereographic projection, one can place knots and links in $\mathbb{R}^3$.

Two knots are equivalent if one can be continuously deformed into another without tearing their strands.



Examples of knots[1]



[1] Rolfsen's knot table, www.katlas.org

Knot invariants:

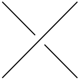
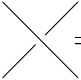
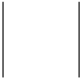
- Alexander polynomial
- Jones polynomial
- HOMFLY-PT polynomial
- Reshetikhin-Turaev invariants

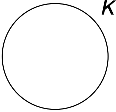
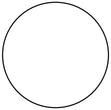
They are closely related to representation theory of quantum groups and Hecke algebras. Applications:

- To distinguish knots
- Alexander polynomial gives bounds for the genus of a surface bounding knot in S^3 (Seifert genus)
- HOMFLY-PT polynomial gives bounds for the braid index

HOMFLY polynomial

HOMFLY-PT polynomial was discovered by Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter, and independently by Przytycki and Traczyk. It is It is a rational function in a and q defined by the following *skein relation*:

a  $-a^{-1}$  $= (q - q^{-1})$ 

 $^{k} = \left(\frac{a-a^{-1}}{q-q^{-1}}\right)^{k-1}$,  $= 1$

Link homology

In the last two decades, several link homology theories were introduced. Some (like *Heegaard Floer homology*) have rather complicated definition, but can be computed explicitly for many knots and links. Others (like *Khovanov and Khovanov-Rozansky homology*) have easier combinatorial or algebraic definition, but are very hard to compute.

Applications:

- *Exact* formulas for Seifert genus
- Bounds for the genus of a surface bounding knot in B^4 (slice genus)
- Bounds for unknotting number, splitting number for links...

Where's algebraic geometry?

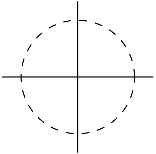
I will talk about two different connections between algebraic geometry and knot theory:

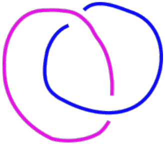
1. Algebraic links and their invariants
2. Hilbert schemes and Khovanov-Rozansky homology

Algebraic links

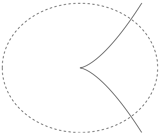
Let $\{f(x,y)=0\}$ be a plane curve singularity in $\mathbb{C}^2$. Its intersection with a small sphere is a knot or link, such links are called *algebraic*.

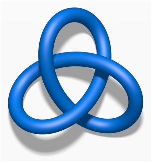
$\{xy=0\}$ corresponds to Hopf link





$\{x^3=y^2\}$ corresponds to trefoil





Algebraic links

Theorem (Milnor,A'Campo,...,Campillo, Delgado, Gusein-Zade)

Alexander polynomial of an algebraic link can be computed in terms of algebraic geometry of $\{f = 0\}$.

Theorem (G.,Némethi)

Heegaard Floer homology of an algebraic link can be computed in terms of algebraic geometry of $\{f = 0\}$.

The answer uses certain strata in the space of (algebraic) functions on $\{f = 0\}$. It is related to multi-dimensional semigroups, hyperplane arrangements,...

Remark

Suppose that you have coins worth 3 *and* 5 *cents. For which* n *you can change* n *cents using these coins? The answer is related to the Alexander polynomial of* $(3, 5)$ *torus knot.*

Algebraic links

Theorem (Maulik, Oblomkov, Shende)

The HOMFLY-PT polynomial of an algebraic link can be computed in terms of algebraic geometry of $\{f = 0\}$*.*

Conjecture (Oblomkov, Rasmussen, Shende)

Khovanov-Rozansky homology of an algebraic link can be computed in terms of algebraic geometry of $\{f = 0\}$*.*

The answer uses *Hilbert scheme of points* on $\{f = 0\}$. The conjecture is proved recently for torus knots, but wide open in general. It is related to deep algebraic geometry, geometric representation theory and combinatorics, and even mathematical physics: compactified Jacobians, Hitchin fibration, affine Springer fibres, Cherednik algebras, Coulomb branches of QFT, q, t-Catalan numbers...

Hilbert schemes

The *symmetric power* of a space X is the space of unordered n-tuples of points on X, denoted by $S^n X$.

For $X = \mathbb{C}$, by Fundamental Theorem of Algebra we have $S^n\mathbb{C} = \mathbb{C}^n$.

For $X = \mathbb{C}^2$, the space $S^n\mathbb{C}^2$ is very singular. However, it has a nice resolution of singularities called $\mathrm{Hilb}^n\,\mathbb{C}^2$, the Hilbert scheme of n points on $\mathbb{C}^2$.

The Hilbert scheme plays a central role in modern algebraic geometry and geometric representation theory.

Hilbert schemes

Conjecture (G.,Neguț,Rasmussen)

The categorical (co)center of the braid group on n strands agrees with the derived category of $\mathrm{Hilb}^n\,\mathbb{C}^2$.

More concretely, we expect that for any braid β there is a vector bundle (or a sheaf, or a complex of sheaves...) $\mathcal{F}_\beta$ on $\mathrm{Hilb}^n\,\mathbb{C}^2$ such that the space of sections of $\mathcal{F}_\beta$ agrees with Khovanov-Rozansky homology of β.

For many braids, we know $\mathcal{F}_\beta$ explicitly.

Hilbert schemes

Even more concretely, recall that the center of the braid group is generated by the *full twist* $\mathbf{ft}$. The closure of $\mathbf{ft}^k$ is the (n, kn) torus link with n components.
Consider the diagonal action of S_n on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.
Let A be the subspace of antisymmetric polynomials, and let J be the ideal generated by A.
The ideal J is closely related to the *homogeneous coordinate ring* of $\mathrm{Hilb}^n \mathbb{C}^2$.

Theorem (G., Hogancamp)

- *For all* $k \geq 0$ *one has* $\mathrm{KhR}(\mathbf{ft}^k) \simeq J^k/(y)J^k$
- *This isomorphism agrees with the muliplication* $\mathbf{ft}^k \times \mathbf{ft}^l = \mathbf{ft}^{k+l}$
- *We define a deformation ("y-ification") of Khovanov-Rozansky homology and show that the deformed homology of* $\mathbf{ft}^k$ *recovers* J^k.

Open questions

There are lots of open questions, and very active recent developments. Here are just a few directions:

- Heegaard Floer homology for non-algebraic links with several components
- Computing Khovanov-Rozansky homology and Hilb$(\{f = 0\})$ beyond torus knots
- Algebro-geometric models for Jones polynomial and Khovanov homology
- Invariants of 3-manifolds
- Categorified skein algebras

Students

Oscar Kivinen (Caltech/U Toronto) is working on affine Springer theory and Hilbert schemes on singular curves

Beibei Liu (Max Planck/Georgia Tech) is working on Heegaard Floer homology and geometric group theory

Advice

- Start going to research seminars as early as possible
- Start reading groups with faculty and postdocs
- It takes 2-3 years to learn the context before you start working on your own problem
- Go to summer schools and conferences. Meet and talk to grad students at other universities!
- **Go to Anna Beliakova's talk today!**

Thank you