Hilbert schemes and link homology (joint with M. Hogancamp, A. Neguț, J. Rasmussen)

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Introduction

In the last two decades, several link homology theories were introduced. Some (like *Heegaard Floer homology*) have rather complicated definition, but can be computed explicitly for many knots and links. Others (like *Khovanov and Khovanov-Rozansky homology*) have easier combinatorial or algebraic definition, but are very hard to compute.
I will talk about the project of computing Khovanov-Rozansky homology by means of algebraic geometry. Rougly speaking:

- To a link L one can associate a vector bundle (or a coherent sheaf) $\mathcal{F}_L$ on the *Hilbert scheme of points on the plane*.
- The space of sections of $\mathcal{F}_L$ (or sheaf cohomology) matches Khovanov-Rozansky homology of L

Hilbert schemes

Hilbert scheme of points

The *symmetric power* $S^n\mathbb{C}^2$ is the space of unordered n-tuples of points on $\mathbb{C}^2$.

The *Hilbert scheme* $\mathrm{Hilb}^n\,\mathbb{C}^2$ is the moduli space of ideals $I \subset \mathbb{C}[x, y]$ such that $\dim \mathbb{C}[x, y]/I = n$. Such an ideal is supported on a finite subset of n points in $\mathbb{C}^2$ (with multiplicities), this defines the Hilbert-Chow morphism:

$$HC : \mathrm{Hilb}^n\,\mathbb{C}^2 \to S^n\mathbb{C}^2.$$

Theorem (Fogarty)

$\mathrm{Hilb}^n\,\mathbb{C}^2$ *is a smooth manifold of dimension* $2n$.

Hilbert schemes

Torus action

The natural scaling action of $(\mathbb{C}^*)^2$ lifts to an action on $S^n\mathbb{C}^2$ and on $\mathrm{Hilb}^n\,\mathbb{C}^2$. It has a finite number of fixed points corresponding to monomial ideals.

Example:

y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2
y	xy	x^2y	x^3y	x^4y	x^5y
1	x	x^2	x^3	x^4	x^5

The ideal is generated by y^3, xy^2, x^3y, x^4

Hilbert schemes

Isospectral Hilbert scheme

The isospectral Hilbert scheme X_n is defined as the reduced fiber product

$$\begin{array}{ccc} X_n & \longrightarrow & \mathrm{Hilb}^n(\mathbb{C}^2) \\ \downarrow & & \downarrow \\ (\mathbb{C}^2)^n & \longrightarrow & S^n\mathbb{C}^2 \end{array}$$

It is singular, but has a lot of nice algebro-geometric properties which were used by Haiman in his proof of $n!$ theorem for Macdonald polynomials.

Consider the diagonal action of S_n on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Let A be the subspace of antisymmetric polynomials, and let J be the ideal generated by A.

Theorem (Haiman)

Both $\operatorname{Hilb}^n \mathbb{C}^2$ *and* X_n *carry natural line bundles* $\mathcal{O}(k)$. *For* $k > 0$, *one has*

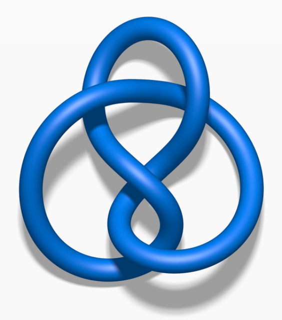
$$H^0(\operatorname{Hilb}^n \mathbb{C}^2, \mathcal{O}(k)) = A^k, \; H^0(X_n, \mathcal{O}(k)) = J^k.$$

In fact, this completely determines $\operatorname{Hilb}^n \mathbb{C}^2$ *and* X_n *as algebraic varieties.*

Knot invariants

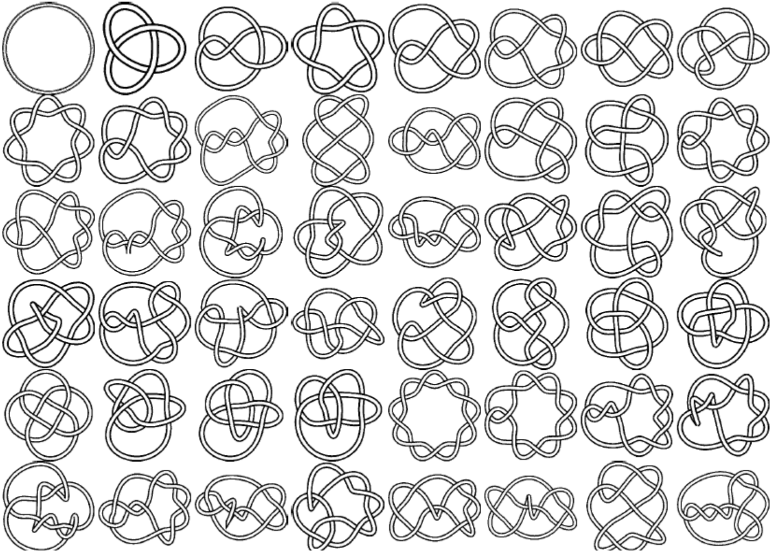
A link is a closed 1-dimensional submanifold in the sphere S^3. If it has one connected component, it is called a knot. Using stereographic projection, one can place knots and links in $\mathbb{R}^3$.

Two knots are equivalent if one can be continuously deformed into another without tearing their strands.



Knot invariants

Examples of knots[1]

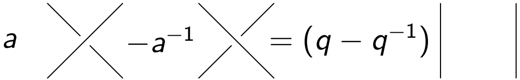


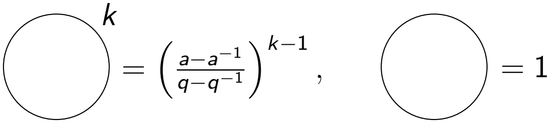
[1] Rolfsen's knot table, `www.katlas.org`

Knot invariants

HOMFLY polynomial

HOMFLY-PT polynomial was discovered by Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter, and independently by Przytycki and Traczyk. It is closely related to representation theory of quantum groups and Hecke algebras. It is a rational function in a and q defined by the following *skein relation*:





Khovanov and Rozansky defined a *categorification* of HOMFLY polynomial using so-called *Soergel bimodules*.

Let $R = \mathbb{C}[x_1, \dots, x_n]$, and

$$B_i = \frac{\mathbb{C}[x_1, \dots, x_n, x'_1, \dots, x'_n]}{x_i + x_{i+1} = x'_i + x'_{i+1}, x_i x_{i+1} = x'_i x'_{i+1}, x_j = x'_j \quad (j \neq i, i+1)}$$

We can regard B_i as a $R - R$-bimodule, where the left action is given by x_i and the right action by x'_i.

Fact: There are bimodule maps $B_i \to R$, $R \to B_i$.

Knot invariants

Khovanov-Rozansky homology

Consider the *complexes* of bimodules:

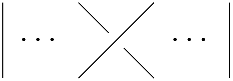
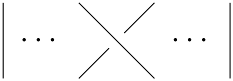
$$T_i = [B_i \to R], \ \ T_i^{-1} = [R \to B_i].$$

Theorem (Rouquier)

The complexes T_i, T_i^{-1} satisfy braid relations up to homotopy. In particular, $T_i \otimes T_i^{-1} \simeq R$.

Corollary

For any braid β on n strands, one can define a complex $T(\beta)$:

$T_i \leftrightarrow$  $T_i^{-1} \leftrightarrow$ 

Knot invariants

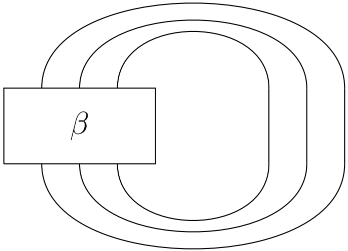
Khovanov-Rozansky homology

We are ready to define Khovanov-Rozansky homology[2]:

$$\mathrm{KhR}(\beta) = H^*(\mathrm{Hom}(R, T(\beta)).$$

Theorem (Khovanov-Rozansky)

$\mathrm{KhR}(\beta)$ *is a topological invariant of the closure of* β*.*



[2] For experts, we focus on the $a = 0$ part only

Knot invariants

Properties

Khovanov-Rozansky homology has lots of nice properties:

- KhR(β) is a triply graded vector space
- In fact, it is a graded module over $R = \mathbb{C}[x_1, \ldots, x_n]$ (more precisely, the variables correspond to link components)
- KhR is functorial: a braid cobordism between β and β' corresponds to a linear map KhR(β) $\to$ KhR(β').
- There is a natural multiplication:

$$\mathrm{KhR}(\alpha) \otimes \mathrm{KhR}(\beta) \to \mathrm{KhR}(\alpha\beta)$$

But it is very hard to compute! Indeed, $T(\beta)$ has 2^c terms for a braid with c crossings...

Knot invariants

Main result

Let $\mathbf{ft}$ denote the full twist braid. The closure of $\mathbf{ft}^k$ is the (n, kn) torus link with n components.

Theorem (G., Hogancamp)

- *For all* $k \geq 0$ *one has* $\mathrm{KhR}(\mathbf{ft}^k) \simeq J^k/(y)J^k$
- *This isomorphism agrees with the muliplication* $\mathbf{ft}^k \times \mathbf{ft}^l = \mathbf{ft}^{k+l}$ *and the* R*-module structure.*
- *We define a deformation ("y-ification") of Khovanov-Rozansky homology and show that the deformed homology of* $\mathbf{ft}^k$ *recovers* J^k*.*

Here J is the Haiman's ideal.

Knot invariants

Construction of the sheaf

Given a braid β, consider the space

$$M_\beta = \oplus_k \mathrm{KhR}(\beta \cdot \mathbf{ft}^k).$$

It is a graded module over the graded algebra

$$\mathcal{A} = \oplus_k \mathrm{KhR}(\mathbf{ft}^k) = \oplus_k J^k/(y)J^k,$$

and therefore defines a sheaf on $\operatorname{Proj} \mathcal{A} = X_n$.

Knot invariants

Example: Torus knots

In many examples, we know the sheaf $\mathcal{F}_\beta$ explicitly. For torus knots:

- (Elias, Hogancamp, Mellit) Khovanov-Rozansky homology of $T(m, n)$ has explicit combinatorial description
- (G., Neguț) It matches the *refined Chern-Simons invariants* of Aganagic-Shakirov and Cherednik
- (G., Mazin, Vazirani) It matches generalized q, t-Catalan numbers of Garsia-Haiman
- Closely related to the *Rational Shuffle Conjecture* in algebraic combinatorics recently proved by Carlsson and Mellit.
- (G., Oblomkov, Rasmussen, Shende, Yun) Related to the geometry of compactified Jacobians and affine Springer fibres

Future directions/work in progress

- Identify the spectral sequences to $sl(N)$ homology
- Oblomkov and Rozansky recently defined another link invariant using sheaves on the Hilbert scheme. Does it agree with $\mathcal{F}_\beta$?
- Soergel bimodules categorify the Hecke algebra. Can we see the algebraic structures (Kazhdan-Lusztig cells, center...) from Hilbert scheme perspective?
- Can we visualize geometric structures on Hilbert schemes (affine charts, monomial ideals, Nakajima correspondences...) using braids?

Thank you