Dyck path algebra and Hilbert schemes

Eugene Gorsky, UC Davis
joint with Erik Carlsson and Anton Mellit

AMS Sectional Meeting, Riverside
November 5, 2017

Outline

- Carlsson and Mellit defined an interesting algebra $\mathbb{B}_{q,t}$
- They also constructed a "polynomial representation" V for this algebra and used it to prove *Shuffle conjecture* in algebraic combinatorics
- We realize this representation geometrically using the equivariant K-theory of certain strata in flag Hilbert schemes of points on the plane
- As a consequence, we find an interesting basis of "generalized Macdonald polynomials" in V corresponding to torus fixed points. The matrix elements of the generators of $\mathbb{B}_{q,t}$ factorize nicely.

The spaces

The *parabolic flag Hilbert scheme* $\mathrm{PFH}_{n,n-k}$ is the moduli space of flags

$$\{I_{n-k} \supset I_{n-k+1} \supset \ldots \supset I_n \supset yI_{n-k}\},$$

where I_j is a codimension j ideal in $\mathbb{C}[x,y]$ for all j.

Example

For $k = 0$ we get $\mathrm{PFH}_{n,n} = \mathrm{Hilb}^n(\mathbb{C}^2)$.

Example

For $k = n$ we have $y \in I_n$, so I_n is scheme-theoretically supported on $\{y = 0\}$. It is easy to see that $\mathrm{PFH}_{n,0} = \mathbb{C}^n$.

In these examples the space $\mathrm{PFH}_{n,n-k}$ is smooth.

The spaces

Theorem

The space $\mathrm{PFH}_{n,n-k}$ *is a smooth manifold of (complex) dimension* $2n - k$*.*

Proof.

Consider $\phi : \mathbb{C}^2 \to \mathbb{C}^2, (x, y) \mapsto (x, y^{k+1})$, define

$$J = \phi^*(I_n) + y\phi^*(I_{n-1}) + \ldots + y^k\phi^*(I_{n-k}) \subset \mathbb{C}[x, y].$$

It is easy to see that $\{I_{n-k}, \ldots, I_n\} \in \mathrm{PFH}_{n,n-k}$ if and only if J is an ideal in $\mathbb{C}[x, y]$. Furthermore, all such ideals J are fixed under the action of $\mathbb{Z}_{k+1}$: $(x, y) \mapsto (x, \zeta y)$. Therefore $\mathrm{PFH}_{n,n-k}$ can be identified with a component in $(\mathrm{Hilb}^N(\mathbb{C}^2))^{\mathbb{Z}_{k+1}}$ and hence is smooth. □

Remark

A similar idea was used by Feigin, Finkelberg, Neguț and Rybikov in their work on affine Laumon spaces.

The spaces

The torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates, this action can be lifted to $\mathrm{PFH}_{n,n-k}$. The fixed points are labeled by the sequences of monomial ideals corresponding to Young diagrams

$$\{\lambda^{(n-k)} \subset \ldots \subset \lambda^{(n)}\}$$

such that the skew shape $\lambda^{(n)} \setminus \lambda^{(n-k)}$ is a (possibly disconnected) *horizontal strip*, that is, has at most one box in each column.

A careful analysis of the above proof of smoothness yields an explicit combinatorial expression for the $(\mathbb{C}^*)^2$ character of the tangent space at a fixed point.

The operators

We construct operators $z_1, \ldots, z_k, T_1, \ldots, T_{k-1}, d_+, d_-$ on the $(\mathbb{C}^*)^2$-equivariant K-theory of PFH.
The operators z_i act by multiplication by line bundles

$$\mathcal{L}_i = I_{n-i}/I_{n-i+1}$$

The operators T_i can be constructed using the space of "partial flags"

$$\mathrm{PFH}^{(i)}_{n,n-k} := \{I_{n-k} \supset \ldots \supset I_{n-i-1} \supset I_{n-i+1} \supset \ldots \supset I_n \supset yI_{n-k}\}.$$

Theorem

The operators T_i and z_i satisfy the relations of the affine Hecke algebra.

Combinatorially, we fix the maximal and minimal Young diagrams $\lambda^{(n-k)} \subset \lambda^{(n)}$ and define an affine Hecke action on the set of standard tableaux of skew shape $\lambda^{(n)} \setminus \lambda^{(n-k)}$.

The operators

The operators d_+ and d_- are more interesting. There are two natural projection maps which forget the first and the last ideal respectively:

$$f : \mathrm{PFH}_{n+1,n-k} \to \mathrm{PFH}_{n,n-k},\ \ g : \mathrm{PFH}_{n,n-k} \to PFH_{n,n-k+1}.$$

We will denote $d_- = g_*, d_+ = q^k(q-1)f^*$.

Define $U_k = \bigoplus_n K_{(\mathbb{C}^*)^2}(\mathrm{PFH}_{n,n-k})$. Then we get operators:

$$z_1, \ldots, z_k, T_1, \ldots, T_{k-1} : U_k \to U_k,$$

$$d_+ : U_k \to U_{k+1}, d_- : U_k \to U_{k-1}.$$

Observe that $U_0 = \bigoplus_n K_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^n(\mathbb{C}^2))$ is the Fock space.

The relations

Let $\varphi = \frac{1}{q-1}[d_+, d_-]$. Then the following relations hold:

$$(T_i - 1)(T_i + q) = 0,\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_i T_j = T_j T_i \quad (|i-j| > 1),$$

$$T_i^{-1} z_{i+1} T_i^{-1} = q^{-1} z_i \ (1 \le i \le k-1),$$

$$z_i T_j = T_j z_i \ (i \notin \{j, j+1\}),\ z_i z_j = z_j z_i \ (1 \le i, j \le k),$$

$$d_-^2 T_{k-1} = d_-^2,\ d_- T_i = T_i d_- \ (1 \le i \le k-2),$$

$$T_1 d_+^2 = d_+^2,\ d_+ T_i = T_{i+1} d_+ \ (1 \le i \le k-1),$$

$$q\varphi d_- = d_- \varphi T_{k-1}, \quad T_1 \varphi d_+ = q d_+ \varphi,$$

$$z_i d_- = d_- z_i, \quad d_+ z_i = z_{i+1} d_+,$$

$$z_1(q d_+ d_- - d_- d_+) = qt(d_+ d_- - d_- d_+) z_k.$$

More operators

Define operators

$$y_i = q^{i-k} T_{i-1}^{-1} \cdots T_1^{-1} \varphi T_{k-1} \cdots T_i$$

One can use the above commutation relations between T_i and ϕ to prove the following:

Lemma

The operators $y_1, \ldots, y_k$, $T_1, \ldots T_{k-1}$ satisfy the relations of the affine Hecke algebra:

$$T_i y_{i+1} T_i = q y_i \ (1 \leq i \leq k-1),$$

$$y_i T_j = T_j y_i \ (i \notin \{j, j+1\}), \ y_i y_j = y_j y_i \ (1 \leq i, j \leq k),$$

Altogether, z_i, T_i and y_i define an action of the braid group of the punctured torus on U_k.

Comparison of representations

Let Λ denote the space of symmetric functions in infinitely many variables. Let $\mathbb{B}_{q,t}$ denote the algebra generated by z_i, T_i, d_+, d_- subject to all above relations.

Theorem

(Carlsson-Mellit) Let $V_k = \Lambda \otimes \mathbb{C}[y_1, \ldots, y_k]$, $V = \oplus_{k=0}^{\infty} V_k$. Then there is a representation of the algebra $\mathbb{B}_{q,t}$ in V such that

$$z_1, \ldots, z_k, T_1, \ldots, T_{k-1} : V_k \to V_k,$$

$$d_+ : V_k \to V_{k+1}, d_- : V_k \to V_{k-1}.$$

In this representation, y_i act as multiplication operators, and T_i act as Demazure-Lusztig operators.

Theorem

There is an isomorphism $U_k \simeq V_k$. The corresponding representations of $\mathbb{B}_{q,t}$ are isomorphic up to a twist by a certain endomorphism of the algebra.

Application: Elliptic Hall algebra

Schiffmann-Vasserot and Feigin-Tsymbaliuk constructed an action of the *elliptic Hall algebra* on the equivariant K-theory of Hilbert schemes (that is, on U_0). The key operators were constructed from simple Nakajima correspondences

$$P_{1,k}(-) = q_*(\mathcal{L}^k \otimes p^*(-)),$$

$$\mathrm{Hilb}^n(\mathbb{C}^2) \xleftarrow{p} \mathrm{Hilb}^{n,n+1}(\mathbb{C}^2) \xrightarrow{q} \mathrm{Hilb}^{n+1}(\mathbb{C}^2).$$

Observe that $\mathrm{Hilb}^{n,n+1}(\mathbb{C}^2) = \mathrm{PFH}_{n+1,n} \times \mathbb{C}$ and

$$P_{1,k} = d_- z_1^k d_+ : U_0 \to U_0.$$

Other generators of the elliptic Hall algebra can be described as

$$P_{m,n} = d_- z_1^{S_1} y_1 z_1^{S_2} y_1 \cdots z_1^{S_m} d_+,$$

where $S_i = \lfloor \frac{in}{m} \rfloor - \lfloor \frac{(i-1)n}{m} \rfloor$. Using the geometric realization of the operators, we can describe $P_{m,n}$ by an explicit correspondence which agrees with the work of Negut.

Application: Elliptic Hall algebra

The combinatorial *Shuffle conjecture* (proven by Carlsson-Mellit) and its "rational" version (proven by Mellit) yield expressions

$$P_{m,n}(1) = \sum_{\pi} \chi_\pi,$$

where π is a lattice path in $m \times n$ rectangle above the diagonal, and χ_π is a certain symmetric function associated to π. The rough sketch of their proof is:

- Write χ_π as an explicit combination of d_+ and d_- applied to $1 \in U_0$
- Prove that this combination agrees with $P_{m,n}(1)$ using the commutation relations in $\mathbb{B}_{q,t}$.

Work in progress: find a geometric interpretation of χ_π for all π.

Thank you