Stable bases and q-Fock space

Eugene Gorsky, Andrei Neguț

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Stable bases: properties

I will describe a new class of symmetric functions S_λ^c introduced in the work of Maulik and Okounkov on equivariant K-theory of Hilbert schemes of points. Here λ is a partition and c is a real number. They satisfy the following properties:

- For a fixed c, $\{S_\lambda^c\}$ is a basis in $\Lambda(q,t)$. The degree of S_λ^c equals $|\lambda|$.
- For any c, c' the change of basis between S_λ^c and $S_\lambda^{c'}$ is triangular in dominance order.
- In a fixed degree $n = |\lambda|$ the basis S_λ^c is piecewise constant in c. That is, there is a discrete set of "walls" (depending on n), the basis is constant between walls and changes as one crosses a wall.
- At slope $c = 0$ we get plethystically modified Schur functions: $S_\lambda^0 = s_\lambda[p_k \mapsto \frac{p_k}{1-q^k}]$
- In the limit $c = \infty$ we get modified Macdonald polynomials: $S_\lambda^\infty = \widetilde{H}_\lambda$

Stable bases: properties

Bergeron and Garsia introduced an important operator $\nabla : \Lambda(q,t) \to \Lambda(q,t)$ which is diagonal in the modified Macdonald basis:

$$\nabla \widetilde{H}_\lambda = \left(\prod_{(x,y)\in\lambda} q^x t^y \right) \widetilde{H}_\lambda .$$

Theorem

The shift of the slope by 1 corresponds to the action of ∇: $S_\lambda^{c+1} = \nabla S_\lambda^c$.

Corollary

For integer slopes, the stable basis can be computed explicitly:

$$S_\lambda^k = \nabla^k S_\lambda^0 = \nabla^k s_\lambda \left[p_k \mapsto \frac{p_k}{1-q^k} \right], \quad k \in \mathbb{Z}.$$

Stable bases: properties

One can check that the stable bases (in fixed degree n) have natural wall and alcove structure: as one varies the parameter c, the basis is locally constant and changes only at certain "walls". The total number of walls is infinite, but it is finite on each finite interval. Suppose that $c = a/b, GCD(a,b) = 1$.

Fact

The change of basis from $c - \varepsilon$ to $c + \varepsilon$ has a block triangular form: two partitions λ and μ belong to the same block if and only if λ and μ have the same b–core.

Corollary

If there is a wall at c then $b \leq n$.

Elliptic Hall Algebra action

Burban and Schiffmann introduced the Elliptic Hall Algebra, which has generators $P_{m,n}$ for $(m,n) \in \mathbb{Z}^2$. This algebra is known to act on $\Lambda(q,t)$. In particular, up to normalization, $P_{m,0}$ is the multiplication operator by the power sum p_m while $P_{0,n}$ is diagonal in the Macdonald basis:

$$P_{0,n}\widetilde{H}_\lambda = \left(\sum_{(x,y)\in\lambda} q^{kx} t^{ky} \right) \widetilde{H}_\lambda.$$

Also, $\nabla P_{m,n} \nabla^{-1} = P_{m,n+m}$. The action of other $P_{m,n}$ can be computed using the commutation relations in the algebra, however their explicit construction is not so easy. Neguț has obtained explicit formulas for $P_{m,n}$ in the modified Macdonald basis using contour integrals and sums over standard tableaux.

Elliptic Hall Algebra action

In recent years, many interesting connections between this algebra, double affine Hecke algebras, algebraic combinatorics, Hilbert schemes and even knot theory were found.

Theorem (Mellit)

For m and n coprime $P_{m,n} \cdot 1$ is Schur positive and can be described by an explicit combinatorial sum over parking functions in the $m \times n$ rectangle. For $m = n + 1$ this is equivalent to the "Shuffle Theorem" of Carlsson and Mellit.

Assume that m and n are coprime.

Theorem

a) [G., Neguț] The polynomial $(P_{m,n} \cdot 1, e_n)$ equals the "refined Chern-Simons invariant" of the (m, n) torus knot defined by Aganagic-Shakirov and Cherednik.
b) [Hogancamp; Mellit] The polynomial $(P_{m,n} \cdot 1, e_n)$ equals the Poincaré polynomial of the Khovanov-Rozansky homology of the (m, n) torus knot.

Elliptic Hall Algebra action

Assume that m and n are coprime. It turns out that the stable basis with slope $n/m \pm \epsilon$ is behaves well with respect to the action of the operators $P_{km,kn}$:

Theorem

One has

$$P_{m,n} S_{\lambda}^{n/m\pm\epsilon} = \sum_{\mu} M^{\pm}(n, m, \lambda, \mu) S_{\mu}^{n/m\pm\epsilon},$$

where μ is obtained from λ by adding an m-ribbon, and $M^{\pm}(n, m, \lambda, \mu)$ is an explicit monomial in q, t. Similarly, $P_{km,kn}$ adds k-stacks of m-ribbons.

Corollary

$P_{m,n} \cdot 1$ can be written as an alternating sum of $S_{\lambda}^{n/m\pm\epsilon}$, where λ is an m-hook, with some monomial coefficients.

q-Fock space

To give a more precise description of the transition matrix, we need to recall some facts about the q–Fock space. It has a basis $|\lambda\rangle$ labeled by partitions and carries the commuting actions of $U_q\widehat{sl_b}$ and the q–Heisenberg algebra. The $\widehat{sl_b}$ generators f_i (resp. e_i) add (resp. remove) boxes of content i mod b to λ, with weights given by certain powers of q. The q-Heisenberg generators P_i add collections of b–ribbons, also with certain q-weights. There exists a unique q–*antilinear* involution such that

$$\overline{|\rangle} = |\rangle, \ \overline{f_i v} = f_i \overline{v}, \ \overline{P_i v} = P_i \overline{v}.$$

Following Leclerc and Thibon, we define a matrix $A_b(q)$ by the equation

$$\overline{|\mu\rangle} = \sum_{\lambda} a_{\lambda\mu}(q) |\lambda\rangle.$$

q-Fock space

Theorem (Leclerc, Thibon)

The matrix $A_b(q) = (a_{\lambda\mu})$ *has the following properties:*
a) $A_b(q)A_b(q^{-1}) = Id$
b) $A_b(q = 1) = Id$
c) $a_{\lambda,\mu} = 0$ *unless* $|\lambda| = |\mu|$, λ *and* μ *have the same* b*–core and* $\lambda \preceq \mu$
d) $a_{\lambda,\lambda} = 1$
e) $A_b(q) = G_b(q)G_b(q^{-1})^{-1}$, *where* $G_b(q)$ *is the transition matrix between the standard basis and the canonical basis*

Example

For $b = 2$, $|\lambda| = |\mu| = 3$ one has

$$A_2(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & 0 & 1 \end{pmatrix}.$$

Main conjecture

Suppose that, as above, $c = a/b$, $GCD(a, b) = 1$.

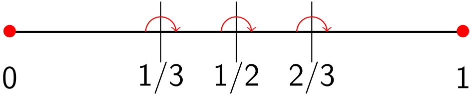
Conjecture

The transition matrix between stable bases at slopes $c - \varepsilon$ and $c + \varepsilon$ equals

$$T_{c-\varepsilon}^{c+\varepsilon}(q, t) = D_c(q, t) A_b(q^b t^b) D_c^{-1}(q, t),$$

where A_b is the Leclerc-Thibon matrix (it depends only on the product qt and the denominator b), and $D_c(q, t)$ is a diagonal matrix with certain explicit monomials in q and t on diagonal.

Example: degree 3



The transition matrix between stable bases at slopes $c = 0$ and $c = 1$ can be computed in two different ways:

1. $S^1_\lambda = \nabla \cdot S^0_{\lambda}$. The operator ∇ is diagonal in modified Macdonald basis, so one needs to relate S^0_λ and $\widetilde{H}_\lambda$.
2. $$S^1 = T^{2/3+\varepsilon}_{2/3-\varepsilon} \circ T^{1/2+\varepsilon}_{1/2-\varepsilon} \circ T^{1/3+\varepsilon}_{1/3-\varepsilon} S^0.$$
$$T^{2/3+\varepsilon}_{2/3-\varepsilon} \sim T^{1/3+\varepsilon}_{1/3-\varepsilon} \sim \begin{pmatrix} 1 & 0 & 0 \\ q - q^{-1} & 1 & 0 \\ q^{-2} - 1 & q - q^{-1} & 1 \end{pmatrix};$$
$$T^{1/2+\varepsilon}_{1/2-\varepsilon} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & 0 & 1 \end{pmatrix}.$$

Main conjecture

Equivalently, this conjecture can be reformulated as following:

Conjecture

For each slope m/n there is an action of $U_q\widehat{sl_n}$ on $\Lambda(q,t)$ such that:

- *It commutes with the action of the q-Heisenberg subalgebra in the elliptic Hall algebra generated by $P_{km,kn},\ k \in \mathbb{Z}$.*
- *The action of $U_q\widehat{sl_n}$ in the stable bases $S_\lambda^{m/n\pm\epsilon}$ agrees (up to a conjugation by an explicit monomial diagonal matrix) with the Leclerc-Thibon action on the q-Fock space.*

Stable bases: definition

The definition is motivated by division with remainder for polynomials. Suppose that $f(x)$ and $g(x)$ are polynomials, then one has a unique decomposition:

$$f(x) = q(x)g(x) + r(x),\ 0 \leq \deg r(x) < \deg g(x).$$

What if f and g are *Laurent* polynomials?

Stable bases: definition

The definition is motivated by division with remainder for polynomials. Suppose that $f(x)$ and $g(x)$ are polynomials, then one has a unique decomposition:

$$f(x) = q(x)g(x) + r(x), \ 0 \leq \deg r(x) < \deg g(x).$$

What if f and g are *Laurent* polynomials?

Lemma

Let f and g be two Laurent polynomials. For any real number c there is a unique decomposition:

$$f(x) = q_c(x)g(x) + r_c(x),$$

$$\mathit{Supp}\ r(x) \subset [\min\deg g(x), \max\deg g(x)) + c.$$

Note that $q_c(x), r_c(x)$ are piecewise constant in c and changes when c crosses an integer "wall".

Stable bases: definition

The stable basis S^c_λ is uniquely defined by the following:

- $(S^c_\lambda, \widetilde{H}_\mu) = 0$ unless $\mu \preceq \lambda$
- $(S^c_\lambda, \widetilde{H}_\lambda) = \prod_{\square\in\lambda}(q^{l(\square)} - t^{a(\square)+1})$
- $\mathrm{Supp}_d(S^c_\lambda, \widetilde{H}_\mu) \subset \mathrm{Supp}_d \prod_{\square\in\mu}(q^{l(\square)} - t^{a(\square)+1})$ $+c(\kappa(\mu) - \kappa(\lambda))$.

Here $(-,-)$ is the Macdonald inner product, $a(\square)$ and $l(\square)$ denote the arm and the leg of $\square$, $\kappa(\lambda)$ is the sum of contents of boxes in λ, and $\mathrm{Supp}_d(F(q,t))$ denotes the projection of the convex hull of the set of nonzero monomials in $F(q,t)$ to the diagonal.

Thank you