Stable bases and q-Fock space

Eugene Gorsky, Andrei Neguț
Columbia University

Representation Theory and Geometry of Symplectic Resolutions
May 19, 2015

Symplectic resolutions

Definition

A conical symplectic resolution is the following collection of data:

1) X - smooth, symplectic algebraic variety, ω - closed, nondegenerate algebraic 2-form on X

2) Action of a torus $T = \mathbb{C}_q^* \times A$ on X such that A preserves the form ω and $\mathbb{C}_q^*$ dilates it: $a^*\omega = \omega, \; s^*\omega = s^2\omega$

3) Affinization $X_0 = \text{Spec } \mathbb{C}[X]$

It is required that X_0 is a cone and $\mathbb{C}_q^*$ contracts it to a point. Furthermore, the natural map $X \to X_0$ needs to be a projective resolution of singularities.

Hilbert schemes

Hilbert scheme of n points on $\mathbb{C}^2$ is an interesting example of a symplectic resolution.

It is known to be smooth, $2n$–dimensional and algebraic symplectic. There is a natural scaling action of 2-dimensional torus on $\mathbb{C}^2$:

$$T = \mathbb{C}_q^* \times \mathbb{C}_t^*, \ (x, y) \mapsto (qtx, qt^{-1}y).$$

Note that $\mathbb{C}_q^*$ dilates the symplectic form on $\mathbb{C}^2$ and $\mathbb{C}_t^*$ preserves this form. This action can be lifted to the Hilbert scheme, where it behaves in a similar way. Finally, the affinization of $\mathrm{Hilb}^n \mathbb{C}^2$ is $S^n\mathbb{C}^2$, and the Hilbert-Chow map is a resolution of singularities.

The fixed points of the action of $A = \mathbb{C}_t^*$ (and of the whole T) on $\mathrm{Hilb}^n \mathbb{C}^2$ are the monomial ideals, naturally labeled by the partitions of n.

K-theoretic stable bases

In a work in progress, Maulik and Okounkov define K-theoretic stable bases for symplectic resolutions. The definition depends on the choice of generic one-parameter subgroup $\sigma : \mathbb{C}^* \to A$ and of a rational line bundle $L \in \mathrm{Pic}_T(X) \otimes \mathbb{Q}$.

The attracting correspondence is defined as $Z^\sigma = \{(x,y) : \lim_{t\to 0} \sigma(t)\cdot x = y\} \subset X \times X^A$. Now the stable basis map $\mathrm{Stab}_L^\sigma : K_T(X^A) \to K_T(X)$ is defined by the following conditions:

a) $\mathrm{Stab}_L^\sigma |_{F\times F'} = 0$ unless $F' \preceq F$

b) $\mathrm{Stab}_L^\sigma |_{F\times F} = \mathcal{O}_{Z^\sigma}|_{F\times F}$

c) $P_A(\mathrm{Stab}_L^\sigma |_{F\times F'}) \subset P_A(\mathcal{O}_{Z^\sigma}|_{F'\times F'}) + L_{F'} - L_F$,

where F, F' denote connected components of X^A and P_A denotes the projection of the set of T-weights to $\mathfrak{a}^\vee$.

K-theoretic stable bases

Since $\mathrm{Pic}(Hilb^n\mathbb{C}^2) = \mathbb{Z}$ for all n, for the Hilbert scheme one can parametrize L by a single rational number c. To simplify notations, we write $S^c_\lambda := \mathrm{Stab}^\sigma_c(\lambda)$. Basic properties:

a) Stable bases for $\mathrm{Hilb}^n\,\mathbb{C}^2$ exist for all c

b) Shift by an integer corresponds to a twist by $\mathcal{O}(1)$:

$$S^{c+1}_\lambda = \mathcal{O}(1) \cdot S^c_\lambda$$

c) At slope $c = 0$ we get plethystically modified Schur functions: $S^0_\lambda = s_\lambda[p_k \mapsto \frac{p_k}{1-q^k}]$

d) At slope $c = \infty$ we get classes of fixed points, given by modified Macdonald polynomials: $S^\infty_\lambda = [I_\lambda] = \widetilde{H}_\lambda$

Walls and blocks

One can check that the stable bases have natural wall and alcove structure: as one varies the parameter c, the basis is locally constant and changes only at certain "walls". The total number of walls is infinite, but it is finite on each finite interval. Suppose that $c = a/b, GCD(a, b) = 1$.

Fact

The change of basis from $c - \varepsilon$ to $c + \varepsilon$ has a block triangular form: two partitions λ and μ belong to the same block if and only if λ and μ have the same b–core. More geometrically, the corresponding fixed points should belong to the same connected component of $(\mathrm{Hilb}^n(\mathbb{C}^2))^{\mathbb{Z}_b}$.

Corollary

If there is a wall at c then $b \leq n$.

q-Fock space

To give a more precise description of the transition matrix, we need to recall some facts about the q–Fock space. It has a basis $|\lambda\rangle$ labeled by partitions and carries the commuting actions of $U_q\widehat{sl_b}$ and the q–Heisenberg algebra. The $\widehat{sl_b}$ generators f_i (resp. e_i) add (resp. remove) boxes of content i mod b to λ, with weights given by certain powers of q. The q-Heisenberg generators P_i add collections of b–ribbons, also with certain q-weights. There exists a unique q*–antilinear* involution such that

$$\overline{|\rangle} = |\rangle, \ \overline{f_i v} = f_i \overline{v}, \ \overline{P_i v} = P_i \overline{v}.$$

Following Leclerc and Thibon, we define a matrix $A_b(q)$ by the equation

$$\overline{|\mu\rangle} = \sum_\lambda a_{\lambda\mu}(q)|\lambda\rangle.$$

q-Fock space

Theorem (Leclerc, Thibon)

The matrix $A_b(q) = (a_{\lambda\mu})$ *has the following properties:*
a) $A_b(q)A_b(q^{-1}) = Id$
b) $A_b(q = 1) = Id$
c) $a_{\lambda,\mu} = 0$ *unless* $|\lambda| = |\mu|$, λ *and* μ *have the same* b*–core and* $\lambda \preceq \mu$
d) $a_{\lambda,\lambda} = 1$
e) $A_b(q) = G_b(q)G_b(q^{-1})^{-1}$, *where* $G_b(q)$ *is the transition matrix between the standard basis and the canonical basis*

Example

For $b = 2$, $|\lambda| = |\mu| = 3$ one has

$$A_2(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & 0 & 1 \end{pmatrix}.$$

Main conjecture

Suppose that, as above, $c = a/b, GCD(a, b) = 1$.

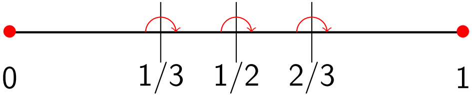
Conjecture

The transition matrix between stable bases at slopes $c - \varepsilon$ and $c + \varepsilon$ equals

$$T_{c-\varepsilon}^{c+\varepsilon}(q, t) = D_c(q, t) A_b(q^b) D_c^{-1}(q, t),$$

where A_b is the Leclerc-Thibon matrix (it depends only on the "scaling parameter" q and the denominator b), and $D_c(q, t)$ is a diagonal matrix with certain explicit monomials in q and t on diagonal.

Example: $\mathrm{Hilb}^3 \mathbb{C}^2$



The transition matrix between stable bases at slopes $c = 0$ and $c = 1$ can be computed in two different ways:

1. $S^1_\lambda = \mathcal{O}(1) \cdot S^0_{\lambda}$. The multiplication by $\mathcal{O}(1)$ is diagonal in fixed point basis, so one needs to relate S^0_λ and $[I_\lambda]$.

2.

$$S^1 = T^{2/3+\varepsilon}_{2/3-\varepsilon} \circ T^{1/2+\varepsilon}_{1/2-\varepsilon} \circ T^{1/3+\varepsilon}_{1/3-\varepsilon} S^0.$$

$$T^{2/3+\varepsilon}_{2/3-\varepsilon} \sim T^{1/3+\varepsilon}_{1/3-\varepsilon} \sim \begin{pmatrix} 1 & 0 & 0 \\ q - q^{-1} & 1 & 0 \\ q^{-2} - 1 & q - q^{-1} & 1 \end{pmatrix};$$

$$T^{1/2+\varepsilon}_{1/2-\varepsilon} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & 0 & 1 \end{pmatrix}.$$

Evidence

1. Confirmed by the explicit computations up to $n = 6$ for all slopes.
2. By construction, $T_{c-\varepsilon}^{c+\varepsilon}$ is triangular, with $1's$ on the diagonal.
3. At $q = 1$, one can prove that $T_{c-\varepsilon}^{c+\varepsilon} = Id$
4. Schiffmann-Vasserot and Feigin-Tsymbaliuk constructed an action of the elliptic Hall algebra $\mathcal{A}$ on $K_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^\bullet)$. For each "slope" $c = a/b$ there is a copy of the q-Heisenberg algebra $\mathcal{H}_c$ inside $\mathcal{A}$. $SL(2,\mathbb{Z})$ acts on $\mathcal{A}$ by automorphisms, permuting these subalgebras in a natural way.

Theorem (Neguț)

The action of $\mathcal{H}_c$ in the stable bases with slopes $c \pm \varepsilon$ is given by adding b–ribbons with certain weights. It is conjugate to the standard degree b action of the q–Heisenberg algebra on the q–Fock space.

Relation to representation theory

The spherical rational Cherednik algebra with parameter c can be considered as a quantization of the Hilbert scheme with a line bundle $c \cdot \mathcal{O}(1)$. It is expected that all above geometric constructions have natural representation theoretic analogues.

Conjecture (Bezrukavnikov, Okounkov)

The bigraded Frobenius characters of standard modules $M_c(\lambda)$ equipped with certain filtrations are equal to S^c_λ.

If one ignores the filtrations, the single-graded characters of $M_c(\lambda)$ do not depend on c, just as $S^c_\lambda(q = 1)$.

Fact

The shift functor $\Phi : \mathcal{O}_c \to \mathcal{O}_{c+1}$ sends Verma modules to Verma modules.

On the Hilbert scheme, Φ matches the multiplication by $\mathcal{O}(1)$.

Relation to representation theory

Let $c = a/b$, $GCD(a, b) = 1$.

Theorem (Rouquier)

The standard modules $M_c(\lambda)$ and $M_c(\mu)$ belong to the same block of the category $\mathcal{O}_c$ if and only if λ and μ have the same b-core.

Theorem (Shan, Vasserot)

There are commuting categorical actions of $\widehat{sl_b}$ and the Heisenberg algebra on $\mathcal{O}_c$.

On the Hilbert scheme, the Heisenberg action is expected to match the one at slope c discussed above.
For example, $P_1^{a/b}|\ \rangle = L_{a/b}(b)$ is a unique finite-dimensional simple representation of the rational Cherednik algebra.

Relation to representation theory

Given a knot in a solid torus, TQFT predicts a Heisenberg algebra action on the Fock space, "creating" this knot, colored by various representations, from the vacuum. These algebras for torus knots $T(a,b)$ are similar to $\mathcal{H}_c$ for $c = a/b$.

Theorem

(a) [G., Oblomkov, Rasmussen, Shende] The character of $L_{a/b}(b)$ coincides with the HOMFLY polynomial of the (a,b) torus knot.
(b) [Etingof, G., Losev] The character of $L_{a/b}(b\lambda)$ coincides with the HOMFLY polynomial of the (a,b) torus knot colored by the Young diagram λ.

Theorem (G., Neguț)

On the Hilbert scheme, the class $P_1^{a/b}|\ \rangle$ coincides with the "refined" HOMFLY polynomial of the (a,b) torus knot, defined by Aganagic-Shakirov and Cherednik.

Thank you